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Important Unitary: Quantum Fourier Transform for Period Finding

QFT_t is an $t \times t$ unitary (acts on t-dim state)

$$QFT_t : |x\rangle \rightarrow \frac{1}{\sqrt{t}} \sum_{y=0}^{-1} e^{2\pi i \frac{x}{t} y} |y\rangle$$

Q: If apply QFT_t to a standard basis state $|x\rangle$ and then measure in standard basis, what is the probability of getting outcome y:

$\boxed{A)\ \frac{1}{t}}$ B) $\frac{1}{\sqrt{t}}$ C) $\frac{xy}{t}$ d $\frac{y}{t}$

Because $\left| \frac{e^{2\pi i \frac{xy}{t}}}{\sqrt{t}} \right|^2 = \left|\frac{1}{\sqrt{t}}\right|^2 \left|e^{2\pi i xy/t}\right|^2 = \frac{1}{t}$

Inverse of QFT

$$QFT_t^{-1} |x\rangle \rightarrow \frac{1}{\sqrt{t}} \sum_{y=0}^{t-1} e^{-2\pi i \frac{x}{t} y} |y\rangle$$

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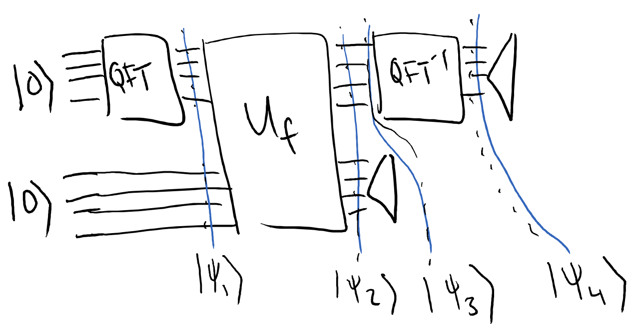
Basic Algorithm:

1. Prepare $|0\rangle_A |0\rangle_B$
   - A: N-dim, B: W-dim
2. Apply QFT_N to A
3. Apply U_f to A, B
4. Measure B in standard basis
5. Apply QFT_N^{-1} to A
6. Measure A in standard basis

Q: Write as circuit 



Total Algorithm

1. Run basic algorithm twice. Get outcomes y, y'.
   Do classical postprocessing on y, y'. Outcome is pretty likely to be $r \rightarrow$ can check if outcome is correct

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- $$\sum_{k=0}^{t-1} e^{\frac{2\pi i k y}{t}} = \sum_{k=0}^{t-1} \left(e^{\frac{2\pi i y}{t}}\right)^k$$

Geometric Series: $\sum_{k=0}^{t-1} r^k = \frac{1-r^{k+1}}{1-r} \quad (r \neq 1)$

$$= \frac{1-e^{\frac{2\pi i t y}{t}}}{1-e^{\frac{2\pi i y}{t}}} = \frac{1-e^{2\pi i y}}{1-e^{2\pi i y/t}} = 0$$

If

- $$\sum_{k=0}^{t-1} a_k \left(\sum_{j=0}^{t-1} b_j |j\rangle\right)$$

$\Downarrow$ Distribute

$$\sum_{k=0}^{t-1} \sum_{j=0}^{t-1} a_k b_j |j\rangle \underset{\text{Swap order}}{\Longrightarrow} \sum_{j=0}^{t-1} \left(\sum_{k=0}^{t-1} a_k b_j\right) |j\rangle$$

amplitude of state $|j\rangle$

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1. $|\psi_1\rangle = \left(QFT|0\rangle\right)|0\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle_A|0\rangle_B$

from exercise

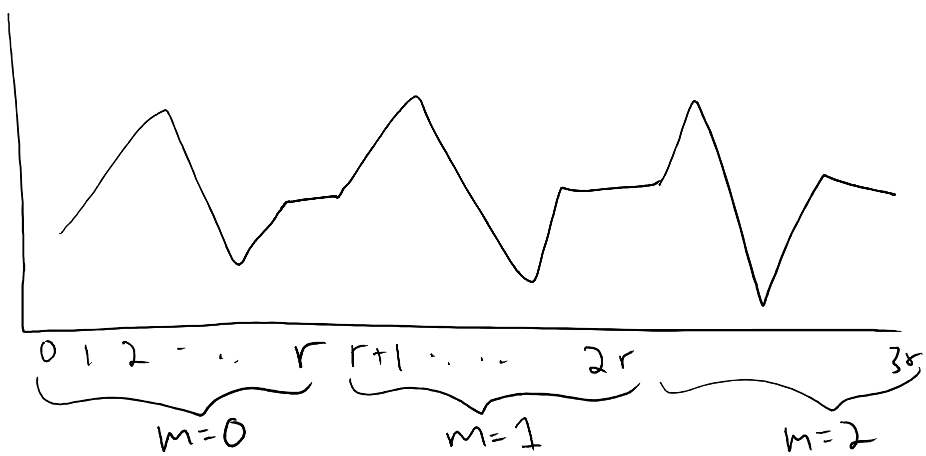
2. $|\psi_2\rangle = \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} U_f|x\rangle|0\rangle = \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1}|x\rangle|f(x)\rangle$

Recall: $f(x)$ is periodic. Let's write $x = mr+b$

period

Q: What is $f(mr+b)$ equal to?

A) $f(r)$ B) $f(m)$ $\boxed{\text{C) } f(b)}$ D) $f(mr)$



$b \in [r]$

$m \in \left[\frac{N}{r}\right]$

$m=i$, $b=j$ corresponds to j^{th} element of i^{th} block of r

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$$|\psi_2\rangle = \frac{1}{\sqrt{N}} \sum_{b=0}^{r-1} \sum_{m=0}^{m_b-1} |mr+b\rangle_A |f(mr+b)\rangle$$

m_b is # blocks where b occurs. If r does not divide N evenly, some values of b will not occur in last block

$$|\psi_2\rangle = \frac{1}{\sqrt{N}} \sum_{b=0}^{r-1} \sum_{m=0}^{m_b-1} |mr+b\rangle |f(b)\rangle$$

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3. Measure B register in standard basis.

To get outcome state, rewrite as

$$|\psi_2\rangle = \sum_i \alpha_i |\phi_i\rangle_A |i\rangle$$

$$|\psi_2\rangle = \sum_{b=0}^{r-1} \left(\frac{1}{\sqrt{N}} \sum_{m=0}^{m_b-1} |mr+b\rangle \right) |f(b)\rangle$$

standard basis states, different for each b by assumption that values are unique within a period

$$\underbrace{\left(\frac{1}{a\sqrt{N}} \times a \sum_{m=0}^{m_b-1} |mr+b\rangle \right)}$$

Q. What is a (approximately) to make this normalized

A) $\frac{1}{\sqrt{N}}$ B) $\frac{1}{\sqrt{b}}$ C) $\frac{1}{\sqrt{m}}$ $\boxed{\text{D) } \sqrt{\frac{r}{N}}}$

Because $m_b = \frac{N}{r}$ or $\frac{N}{r} - 1$

Suppose get outcome $|f(b)=s\rangle$. Let b^* be value s.t. $f(b^*)=s$

After Measurement, state is

$$|\psi_3\rangle = \left(\frac{1}{\sqrt{m_{b^*}}} \sum_{m=0}^{m_{b^*}-1} |mr+b^*\rangle_A \right) |f(b^*)\rangle_B$$

We don't do anything else with B system, since tensor product, can just ignore from this point on