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Basic Quantum Algorithms

Goal:

- Strategies for analyzing quantum algorithms
- Understanding of why quantum algs. can do better than quantum.

Deutsch Algorithm

Consider a one bit function f:

x	$f(x)$
0	?
1	?

Options for f:

x	$f(x)$
0	0
1	0

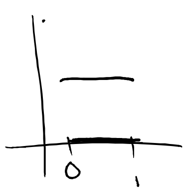
x	$f(x)$
X	1
1	1

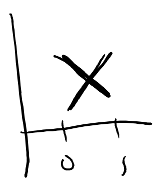
"even"

x	$f(x)$
0	0
1	1

x	$f(x)$
0	1
1	0

"balanced"





Problem: Decide if f is even or balanced.

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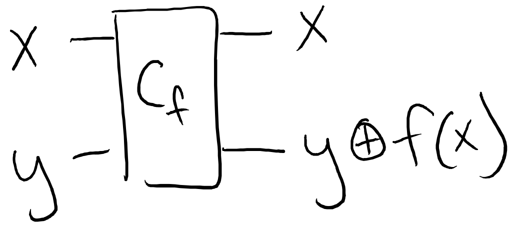
How is f given?

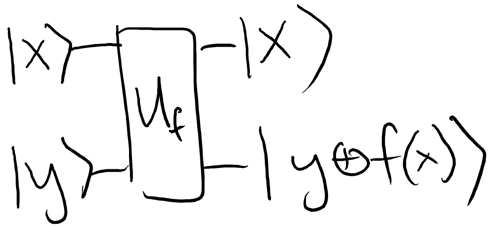
- Classically: $X - \boxed{f} - f(x)$ (non reversible)



(reversible)

$\Rightarrow$ essentially can learn $f(i)$ for any i with one use of $-\boxed{f}-$ or $=\boxed{C_f}=$

- Reversible f (quantum)



$\leftarrow$ $|x\rangle, |y\rangle$ are standard basis states. So if $|x\rangle, |y\rangle$ are standard basis states, acts just like classical.

How $=\boxed{U_f}=$ acts on standard basis states.

* If not standard basis states, U_f does different things

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Problem (precise): How many uses of C_f / U_f are required as part of a classical/quantum circuit to determine if f is balanced or even?

classical answer

quantum answer

Classical Query Complexity

Quantum Query Complexity

Classical query complexity of even/odd = 2

Why: If learn $f(0)$ only, not enough to decide if even or balanced

If learn $f(1)$ only, not enough to decide if even or balanced

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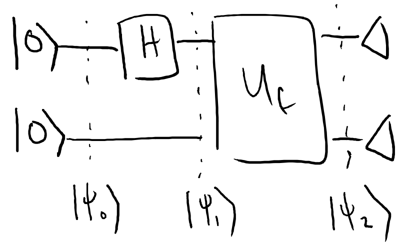
Power (?) of Quantum Superposition



* H creates superposition

$|\psi_0\rangle = |00\rangle$

$|\psi_1\rangle = H|0\rangle \otimes I|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|0\rangle\right)$

$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(U_f|00\rangle + U_f|10\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle|0 \oplus f(0)\rangle + |1\rangle|0 \oplus f(1)\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(|0\rangle|f(0)\rangle + |1\rangle|f(1)\rangle\right)$

standard basis states

Seems great! We are in superposition of two function values! Classically, if apply C_f once, only get info about one value of function

Now measure in standard basis:

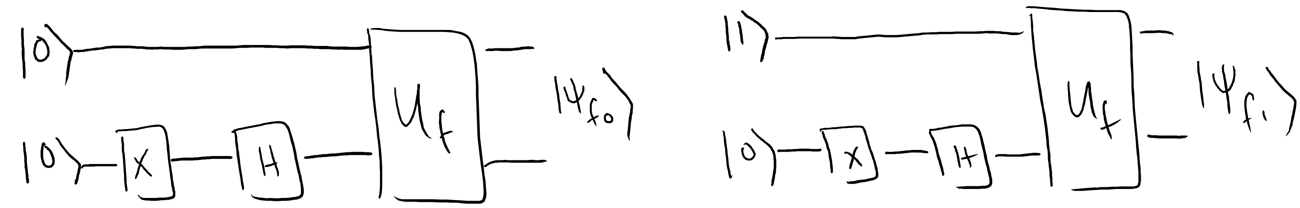
- With prob $1/2$ get outcome $|0\rangle|f(0)\rangle$
- With prob $1/2$ get outcome $|1\rangle|f(1)\rangle$

Recap: $\boxed{H}$ gave superposition over inputs

$\boxed{U_f}$ gave superposition over function input-output pairs

◁ destroyed superposition. Acts just like probabilistic circuit.

Consider the following circuits.



Show $|\psi_{f_0}\rangle = (-1)^{f(0)}|0\rangle|-\rangle$

$|\psi_{f_1}\rangle = (-1)^{f(1)}|1\rangle|-\rangle$

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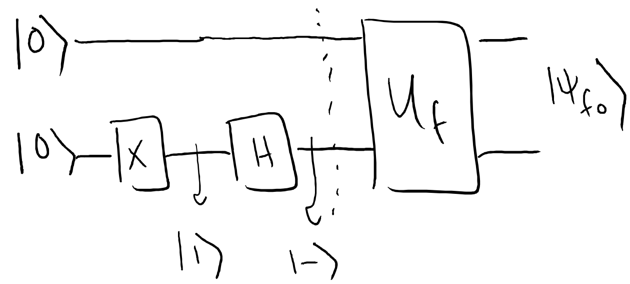
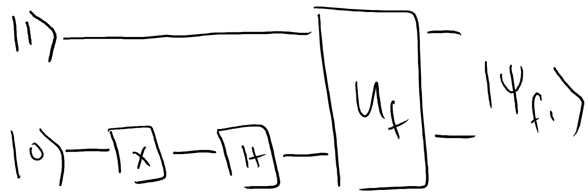
Now try the following:





$$|\psi_{f_0}\rangle = U_f\left(|0\rangle \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$
$$= \tfrac{1}{\sqrt{2}}\left(|0\rangle|0\oplus f(0)\rangle - |0\rangle|1\oplus f(0)\rangle\right)$$
$$= \tfrac{1}{\sqrt{2}}|0\rangle \otimes \left(|0\oplus f(0)\rangle - |1\oplus f(0)\rangle\right)$$

- If $f(0)=0 \Rightarrow \frac{1}{\sqrt{2}}|0\rangle\otimes(|0\rangle - |1\rangle)$
- If $f(0)=1 \Rightarrow \frac{1}{\sqrt{2}}|0\rangle\otimes(|1\rangle - |0\rangle)$

$$= \tfrac{-1}{\sqrt{2}}|0\rangle\otimes(|0\rangle - |1\rangle)$$

$$|\psi_{f_0}\rangle = (-1)^{f(0)}|0\rangle|-\rangle$$

$$U_f|0\rangle|-\rangle = (-1)^{f(0)}|0\rangle|-\rangle$$

$$|\psi_{f_1}\rangle = U_f\left(|1\rangle \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$
$$= \tfrac{1}{\sqrt{2}}|1\rangle\left(|0\oplus f(1)\rangle - |1\oplus f(1)\rangle\right)$$

If $f(1)=0 \Rightarrow \frac{1}{\sqrt{2}}|1\rangle(|0\rangle - |1\rangle)$

If $f(1)=1 \Rightarrow \frac{1}{\sqrt{2}}|1\rangle(|1\rangle - |0\rangle)$

$$|\psi_{f_1}\rangle = (-1)^{f(1)}|1\rangle|-\rangle$$

$$U_f|1\rangle|-\rangle = (-1)^{f(1)}|1\rangle|-\rangle$$

$$\boxed{\begin{aligned} U_f|i\rangle|-\rangle &= (-1)^{f(i)}|i\rangle|-\rangle \\ U_f|i\rangle|+\rangle &= |i\rangle|+\rangle \end{aligned}}$$

Phase Kickback

Can show

$$U_f|0\rangle|+\rangle = |0\rangle|+\rangle$$
$$U_f|1\rangle|+\rangle = |1\rangle|+\rangle,$$

so if 2nd qubit is $|+\rangle$, no phase kickback.

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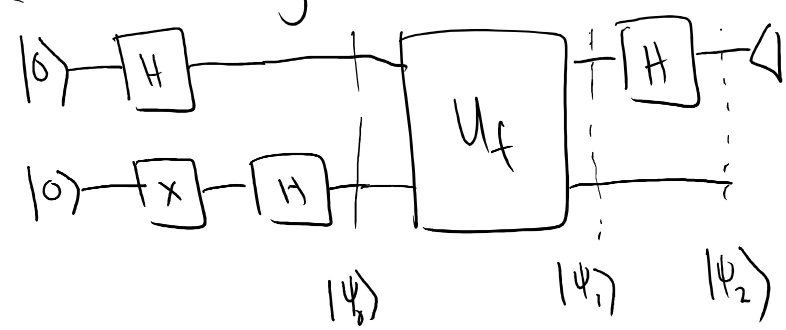
"Phase Kickback"

(-1) is called phase

Seems like U_f does something to 2^{nd} qubit based on first qubit, but actually can be seen as adding phase to first qubit based on 2^{nd} qubit (depending on $|+\rangle$ or $|-\rangle$)

Quantum mechanics is weird because "control" depends on basis choice. In standard basis, U_f control is first qubit, but if 2^{nd} register is in $|+\rangle|-\rangle$ basis, control is second qubit

Deutsch Algorithm



$$|\psi_0\rangle = |+\rangle|-\rangle$$

$$|\psi_1\rangle = U_f|+\rangle|-\rangle = U_f\frac{|0\rangle|-\rangle}{\sqrt{2}} + U_f\frac{|1\rangle|-\rangle}{\sqrt{2}} = (-1)^{f(0)}\frac{|0\rangle|-\rangle}{\sqrt{2}} + (-1)^{f(1)}\frac{|1\rangle|-\rangle}{\sqrt{2}}$$

Pull $(-1)^{f(0)}$ out of both terms : $|\psi_1\rangle = \frac{(-1)^{f(0)}}{\sqrt{2}}\left[|0\rangle|-\rangle + (-1)^{f(0)\oplus f(1)}|1\rangle|-\rangle\right]$

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$$= \frac{(-1)^{f(0)}}{\sqrt{2}} \left(|0\rangle + (-1)^{f(0)\oplus f(1)} |1\rangle \right) |-\rangle$$

Global phase, we can ignore!

- $f(0) \oplus f(1) = 0$ (EVEN!) $\longrightarrow$ $|\psi_1\rangle = (|0\rangle + |1\rangle)|-\rangle = |+\rangle|-\rangle$
- $f(0) \oplus f(1) = 1$ (Balanced!) $\longrightarrow$ $|\psi_1\rangle = (|0\rangle - |1\rangle)|-\rangle = |-\rangle|-\rangle$

If can measure first qubit in $\{|+\rangle, |-\rangle\}$ basis, will get outcome $|+\rangle$ if even, outcome $|-\rangle$ if odd!

We can only measure in standard basis :(

$-\boxed{H}-$ converts between $\{|0\rangle, |1\rangle\} \longleftrightarrow \{|+\rangle, |-\rangle\}$

Final Step

$$|\psi_2\rangle = H \otimes I |\psi_1\rangle$$

$$\left.\begin{array}{l} H\otimes I |+\rangle|-\rangle = |0\rangle|-\rangle \\ H\otimes I |-\rangle|-\rangle = |1\rangle|-\rangle \end{array}\right\} |\psi_2\rangle \text{ in even/balanced case}$$

So $-\boxed{H}-\triangleleft$ is measurement in $\{|+\rangle, |-\rangle\}$ basis

Thus circuit decides with certainty if f is even or balanced, using only <u>one</u> query to U_f. Thus

$$\left.\begin{array}{l} \text{Quantum Query Complexity} = 1 \\ \text{Classical Query Complexity} = 2 \end{array}\right\} \text{Quantum Speed-Up!}$$