CS333

1. Suppose you would like to transform a qubit in the state $|0\rangle$ into a qubit in the state $|1\rangle$. There are many unitaries that can accomplish this. Describe the complete set of unitaries that accomplish this transformation both in terms of the Bloch sphere and in terms of a matrix representation.

   Solution Any 180° rotation of the Bloch sphere about an axis that lies on the equator will accomplish the rotation.

   We can write the U in matrix form as:

   $$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1}$$

   We know that $U|0\rangle = |1\rangle$, so we want

   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

   So $a = 0$, and $c = 1$. But we also know that U must be unitary, so $UU^\dagger = I$. If we take $UU^\dagger$, we get

   $$\begin{pmatrix} 0 & b \\ 1 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} |b|^2 & b\bar{d} \\ d\bar{b} & 1 + |d|^2 \end{pmatrix} \tag{3}$$

   Thus $d = 0$, and $b = e^{i\phi}$, so the set of unitaries are those of the form

   $$U = \begin{pmatrix} 0 & e^{i\phi} \\ 1 & 0 \end{pmatrix}. \tag{4}$$

2. Prove that

   $$C_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{5}$$

   is an entangling gate. By entangling gate, I mean it is possible to have this unitary act on an unentangled state, and have the result be an entangled state.

Solution You can verify that $C_p(|+\rangle|-\rangle)$ is entangled.

3. (This problem has a lot more calculation than I would have you do on the exam, but it is good practice.) Here is another quantum game. This time with three players, Alice, Bob and Charlie. The referee sends a bit x to Alice, y to Bob, and z to Charlie. Alice returns a bit a, Bob a bit b, and Charlie a bit c. The referee either gives all players 0, or the referee gives two players 1's and one player a 0. The players can not communicate. They win if $a \oplus b \oplus c = x \vee y \vee z$. Here is a table of the winning conditions:

xyz	$a \oplus b \oplus c$
000	0
011	1
101	1
110	1

(6)

   (a) Describe a deterministic (non-probabilistic) classical strategy that has as large a winning probability as possible (averaged over the choice of x, y, and z) and determine what the chance of winning is. (Probabilistic strategies can't do better than deterministic strategies for these games, so restricting to deterministic strategies doesn't hurt us.)

   (b) Suppose Alice, Bob, and Charlie share the 3-qubit quantum state $|\psi\rangle = \frac{1}{2}|000\rangle - \frac{1}{2}|011\rangle - \frac{1}{2}|101\rangle - \frac{1}{2}|110\rangle$. Prove that this state is entangled. That is, prove that there is no tensor product of three single qubit states that equals $|\psi\rangle$.

   (c) Consider the strategy that if a player receives a 0, they measure using the basis $\{|0\rangle, |1\rangle\}$, and return 0 with outcome $|0\rangle$ and 1 with outcome $|1\rangle$, if a player receives a 1, they measure using the basis $\{|+\rangle, |-\rangle\}$, and return 0 with outcome $|+\rangle$ and 1 with outcome $|-\rangle$

      i. If $x = y = z = 0$, what is their success probability?

      ii. If $x = 0$, $y = z = 1$, what is their success probability? (By symmetry, this case is the same as the remaning cases.)

Solution

   (a) The maximum success probability is 3/4, which can be achieved if Alice, Bob, and Charlie always return 1, no matter what bit they see.

   (b) If the state is not entangled, there exist $e, f, g, h, i, j \in \mathbb{C}$ such that

$$\begin{pmatrix} e \\ f \end{pmatrix} \otimes \begin{pmatrix} g \\ h \end{pmatrix} \otimes \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} eg \\ eh \\ fg \\ fh \end{pmatrix} \otimes \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} egi \\ egj \\ ehi \\ ehj \\ fgi \\ fgj \\ fhi \\ fhj \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} \tag{7}$$

   Now we see $egj = 0$, so $e = 0$, $g = 0$, or $j = 0$. But $e \neq 0$ and $g \neq 0$ because $egi = 1$. But also $j \neq 0$, because $fgj = -1$. Therefore, the state is entangled.

(c) If $x = y = z = 0$, Alice, Bob, and Charlie all do a standard basis measurement, which corresponds to a standard basis measurement ($M = \{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$ on the full 3 qubit state. In this case, they get $|000\rangle$ with probability $1/4$, and so return $a = b = c = 0$ and win, or they get outcome $|011\rangle$ with probability $1/4$, and so return $a = 0$, $b = c = 1$ and win, or they get outcome $|101\rangle$ with probability $1/4$, and so return $b = 0$, $a = c = 1$ and win, or they get outcome $|110\rangle$ with probability $1/4$, and so return $c = 0$, $b = a = 1$ and win. Thus they will win with certainty with this probability in this case.

(d) We calculate the probability for each of the 8 outcomes:

- $$\begin{aligned}&\left|\langle 0++|\left(\frac{1}{2}|000\rangle - \frac{1}{2}|011\rangle - \frac{1}{2}|101\rangle - \frac{1}{2}|110\rangle\right)\right|^2 \\ &= \frac{1}{4}\left|\langle 0|0\rangle\langle +|0\rangle\langle +|0\rangle - \langle 0|0\rangle\langle +|1\rangle\langle +|1\rangle - \langle 0|1\rangle\langle +|0\rangle\langle +|1\rangle - \langle 0|1\rangle\langle +|1\rangle\langle +|0\rangle\right|^2 \\ &= \frac{1}{4}\left|\frac{1}{2} - \frac{1}{2}\right|^2 \\ &= 0,\end{aligned} \tag{8}$$

- $$\begin{aligned}&\left|\langle 0+-|\left(\frac{1}{2}|000\rangle - \frac{1}{2}|011\rangle - \frac{1}{2}|101\rangle - \frac{1}{2}|110\rangle\right)\right|^2 \\ &= \frac{1}{4}\left|\langle 0|0\rangle\langle +|0\rangle\langle -|0\rangle - \langle 0|0\rangle\langle +|1\rangle\langle -|1\rangle\right|^2 \\ &= \frac{1}{4}\left|\frac{1}{2} + \frac{1}{2}\right|^2 \\ &= \frac{1}{4},\end{aligned} \tag{9}$$

  which corresponds to a win.

- $$\left|\langle 0-+|\left(\frac{1}{2}|000\rangle - \frac{1}{2}|011\rangle - \frac{1}{2}|101\rangle - \frac{1}{2}|110\rangle\right)\right|^2 = \frac{1}{4}, \tag{10}$$

  by symmetry with previous case, corresponding to a win.

- $$\begin{aligned}&\left|\langle 0--|\left(\frac{1}{2}|000\rangle - \frac{1}{2}|011\rangle - \frac{1}{2}|101\rangle - \frac{1}{2}|110\rangle\right)\right|^2 \\ &= \frac{1}{4}\left|\langle 0|0\rangle\langle -|0\rangle\langle -|0\rangle - \langle 0|0\rangle\langle -|1\rangle\langle -|1\rangle\right|^2 \\ &= \frac{1}{4}\left|\frac{1}{2} - \frac{1}{2}\right|^2 \\ &= 0,\end{aligned} \tag{11}$$

  which corresponds to a win.

- $$\begin{aligned}&\left|\langle 1++|\left(\frac{1}{2}|000\rangle-\frac{1}{2}|011\rangle-\frac{1}{2}|101\rangle-\frac{1}{2}|110\rangle\right)\right|^2\\&=\frac{1}{4}\left|\langle 1|0\rangle\langle +|0\rangle\langle +|0\rangle-\langle 1|0\rangle\langle +|1\rangle\langle +|1\rangle-\langle 1|1\rangle\langle +|0\rangle\langle +|1\rangle-\langle 1|1\rangle\langle +|1\rangle\langle +|0\rangle\right|^2\\&=\frac{1}{4}\left|-\frac{1}{2}-\frac{1}{2}\right|^2\\&=\frac{1}{4},\end{aligned}\tag{12}$$

  which is a win.

- $$\begin{aligned}&\left|\langle 1+-|\left(\frac{1}{2}|000\rangle-\frac{1}{2}|011\rangle-\frac{1}{2}|101\rangle-\frac{1}{2}|110\rangle\right)\right|^2\\&=\frac{1}{4}\left|-\langle 1|1\rangle\langle +|0\rangle\langle -|1\rangle-\langle 1|1\rangle\langle +|1\rangle\langle -|0\rangle\right|^2\\&=\frac{1}{4}\left|\frac{1}{2}-\frac{1}{2}\right|^2\\&=0,\end{aligned}\tag{13}$$

- $$\left|\langle 1-+|\left(\frac{1}{2}|000\rangle-\frac{1}{2}|011\rangle-\frac{1}{2}|101\rangle-\frac{1}{2}|110\rangle\right)\right|^2=0,\tag{14}$$

  by symmetry

- $$\begin{aligned}&\left|\langle 1--|\left(\frac{1}{2}|000\rangle-\frac{1}{2}|011\rangle-\frac{1}{2}|101\rangle-\frac{1}{2}|110\rangle\right)\right|^2\\&=\frac{1}{4}\left|-\langle 1|1\rangle\langle -|0\rangle\langle -|1\rangle-\langle 1|1\rangle\langle -|1\rangle\langle -|0\rangle\right|^2\\&=\frac{1}{4}\left|\frac{1}{2}+\frac{1}{2}\right|^2\\&=\frac{1}{4}\end{aligned}\tag{15}$$

  which is a win.

We see all outcomes that have non-zero probability are wins, so Alice, Bob, and Charlie always win! This is an example of a game where you can win all of the time with an entangled resource, but not all of the time without the resource.