CS333 - Gates Worksheet

1. We have three ways of representing unitaries: in matrix form, in terms of how it transforms standard basis states, and in ket-bra form. I will give you a unitary in one of the forms; please write its representation using the other two forms, and also verify that it is indeed unitary.

   (a)

$$CP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \tag{1}$$

   (b)

$$\begin{aligned} U|0\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right), \\ U|1\rangle &= \frac{1}{\sqrt{2}}\left(-i|0\rangle - |1\rangle\right) \end{aligned} \tag{2}$$

   (c) $V = |00\rangle\langle 00| + |11\rangle\langle 11| + \frac{1}{\sqrt{2}}\left(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| - |10\rangle\langle 10|\right)$

Solution

(a) In ket-bra form: $CP = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + i|11\rangle\langle 11|$, and

$$\begin{aligned} CP|00\rangle &= |00\rangle \\ CP|01\rangle &= |01\rangle \\ CP|10\rangle &= |10\rangle \\ CP|11\rangle &= i|11\rangle. \end{aligned} \tag{3}$$

To verify that CP is unitary, note that $(CP)^\dagger$ is the same as CP but with i replaced by $-i$. Then you can check that $CP(CP)^\dagger = I$.

(b)

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \tag{4}$$

and $U = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| - |1\rangle\langle 1| - i|0\rangle\langle 1| + i|1\rangle\langle 0|\right)$. Finally in this case $U = U^\dagger$, and you can verify that $UU = I$, so U is unitary.

(c)

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

and

$$\begin{aligned} V|00\rangle =& |00\rangle \\ V|01\rangle =& \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ V|10\rangle =& \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\ V|11\rangle =& |11\rangle. \end{aligned} \tag{6}$$

In this case, $V^\dagger = V$, and you can verify that $VV = I$.

2. If U is a unitary and $|\psi\rangle$ is a quantum state, is $U|\psi\rangle$ always a state? (Think about what our criteria are for quantum states.)

   Solution For $U|\psi\rangle$ to be a quantum state, it must be a normalized complex-valued vector. Since a matrix times a vector gives us another vector, we do have a complex-valued vector, so we just need to check it is is normalized. Let $|\psi'\rangle = U|\psi\rangle$. To check that it is normalized, we need to verify that $\langle\psi'|\psi'\rangle = 1$. But $\langle\psi'| = \langle\psi|U^\dagger$, so

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = \langle\psi|I|\psi\rangle = \langle\psi|\psi\rangle = 1. \tag{7}$$

3. Unitary operations are reversible. This means for any unitary U there is another unitary matrix V that undoes the action of U. What is this other matrix and how do you know it is a unitary (and hence a valid quantum operation)? Unitary operations contrast with quantum measurements, which we describe as being non-reversible. Why are measurements non-reversible?

   Solution $U^\dagger$ undoes the action of U, since $U^\dagger U = I$. Now we just need to check that $U^\dagger$ is unitary. $U^\dagger$ is unitary if $U^\dagger(U^\dagger)^\dagger = I$, but $(U^\dagger)^\dagger = U$, and we already know $U^\dagger U = 1$, so $U^\dagger$ is indeed unitary.

   This contrasts with quantum measurements because of the collapse that occurs during a measurement. If you did not know the details of the state of the system before measurement, the collapse means that information is lost. However, with a unitary operation, even if we don't know what the original state was before the unitary, we can always recover it by applying the conjugate transpose of the unitary.