CS333 - Final Review

1. Consider the following two-qubit state:

$$\frac{1}{\sqrt{3}}\left(|0\rangle|+\rangle + |-\rangle|1\rangle\right) \tag{1}$$

   (a) **[3 points]** If you measure the first qubit of $|\psi\rangle$ using the basis $\{|0\rangle, |1\rangle\}$, what is the probability of getting outcome $|0\rangle$?

   (b) **[3 points]** If you get outcome $|0\rangle$ on the first qubit, what state will the second qubit collapse to?

   (c) **[3 points]** If you measure the first qubit of $|\psi\rangle$ using the basis $\{|+\rangle, |-\rangle\}$, what is the probability of getting outcome $|0\rangle$?

   (d) **[3 points]** If you get outcome $|+\rangle$ on the first qubit, what state will the second qubit collapse to?

Solution

(a) We can rewrite $|\psi\rangle$ as

$$\begin{aligned}|\psi\rangle =& \frac{1}{\sqrt{3}}\left(|0\rangle|+\rangle + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|1\rangle\right)\\ =& \frac{1}{\sqrt{3}}\left(|0\rangle\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{2}{\sqrt{2}}|1\rangle\right) + |1\rangle|something\rangle\right)\end{aligned} \tag{2}$$

Then we can normalize the state that the second qubit is in if the first qubit is in $|0\rangle$ as

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(\frac{\sqrt{5}}{\sqrt{2}}|0\rangle\left(\frac{\sqrt{2}}{\sqrt{5}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{2}{\sqrt{2}}|1\rangle\right)\right) + |1\rangle|something\rangle\right) \tag{3}$$

Thus the probability of outcome $|0\rangle$ is $\left|\frac{1}{\sqrt{3}} \times \frac{\sqrt{5}}{\sqrt{2}}\right|^2 = 5/6$.

(b) Reading off of the previous equation, we see the second qubit is left in the state

$$\frac{\sqrt{2}}{\sqrt{5}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{2}{\sqrt{2}}|1\rangle\right) \tag{4}$$

(c) We can rewrite the state as

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)|+\rangle + |-\rangle|1\rangle\right) \\ = \frac{1}{\sqrt{6}}|+\rangle|+\rangle + \alpha|-\rangle|something\rangle \tag{5}$$

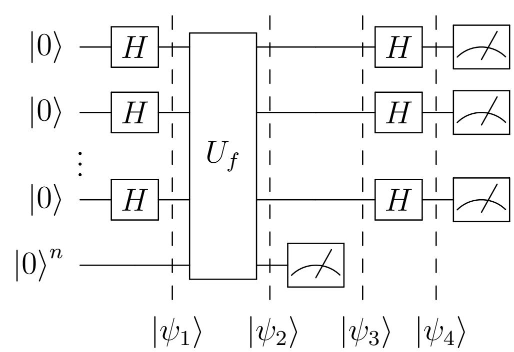
So the probability of getting $|+\rangle$ is 1/6.

(d) The second qubit will be left in the state $|+\rangle$.

2. Suppose you have a function $f : \{0,1\}^n \to \{0,1\}^n$, where $f(x) = f(y)$ if and only if $x = y \oplus s$ for some $s \in \{0,1\}^n$. (Here $\oplus$ means addition mod 2 for each bit of the string.) Otherwise, there is no structure in terms of which input gets assigned to which output. Recall for standard basis states: $H^{\otimes n}|x\rangle = \sum_{y=0}^{2^n-1}(-1)^{x\cdot y}|y\rangle$, where $x \cdot y = \sum_{j=1}^{n} x_j y_j$ where x_j is the jth bit of x and y_j is the jth bit of y.

(a) What is the classical query complexity of determining s?

(b) Suppose you have a unitary that acts as $U_f|x\rangle|0\rangle = |x\rangle|f(x)\rangle$. (For this algorithm, it doesn't matter how U_f acts on other standard basis states.) If we run the following algorithm:



(6)

What are the states at each point?

(c) If you can find $O(n)$ (randomly chosen) z such that $z \cdot s = 0 \mod 2$, then can figure out s. Use this fact to determine the quantum query complexity of learning s.

Solution

(a) We need to find an pair $\{x, y\}$ such that $f(x) = f(y)$. We might have to look at half of the inputs before we find such a pair, so the query complexity is $O(2^n)$ (since the total number of inputs is 2^n.)

(b)

$$|\psi_1\rangle = \frac{1}{2^{n/2}} \sum_{x\in\{0,1\}^n} |x\rangle|0\rangle. \quad (7)$$

$$|\psi_2\rangle = \frac{1}{2^{n/2}} \sum_{x\in\{0,1\}^n} |x\rangle|f(x)\rangle. \quad (8)$$

Suppose we measure b. Then there are exactly two inputs x_b and y_b such that $f(x_b) = f(y_b) = b$, and $x = y_b \oplus s$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|x_b\rangle + |y_b\rangle)|b\rangle. \quad (9)$$

(We drop the B system from here forward.)

$$\begin{aligned}
|\psi_4\rangle &= \frac{1}{\sqrt{2}} \times \frac{1}{2^{n/2}} \sum_{z\in\{0,1\}^n} (-1)^{x_b\cdot z}|z\rangle + (-1)^{y_b\cdot z}|z\rangle \\
&= \frac{1}{\sqrt{2^{n+1}}} \sum_{z\in\{0,1\}^n} ((-1)^{x_b\cdot z} + (-1)^{y_b\cdot z})|z\rangle \\
&= \frac{1}{\sqrt{2^{n+1}}} \sum_{z\in\{0,1\}^n} ((-1)^{x_b\cdot z} + (-1)^{(x_b\oplus s)\cdot z})|z\rangle \\
&= \frac{1}{\sqrt{2^{n+1}}} \sum_{z\in\{0,1\}^n} ((-1)^{x_b\cdot z} + (-1)^{x_b\cdot z + s\cdot z})|z\rangle \\
&= \frac{1}{\sqrt{2^{n+1}}} \sum_{z\in\{0,1\}^n} (-1)^{x_b\cdot z}(1 + (-1)^{s\cdot z})|z\rangle. \quad (10)
\end{aligned}$$

If $s \cdot z \equiv 1 \mod 2$, then we have zero amplitude on that $|z\rangle$, so we will never measure it. Thus we will only measure $|z\rangle$ such that $s \cdot z \equiv 0 \mod 2$.

(c) By repeating this process $O(n)$ times, we will obtain a set of bit strings that have inner product 0 with s, and then we can determine s.

3. I said that there is a similar code to the 3-qubit bit-flip code we looked at in class that corrects Z errors. The idea is to convert from the $|0\rangle/|1\rangle$ view to a $|+\rangle/|-\rangle$ view, since $|+\rangle/|-\rangle$ are sensitive to Z errors, but the approach otherwise should be similar to the 3-qubit bit-flip code.

   (a) Draw a circuit that encodes a qubit $a|0\rangle + b|1\rangle$ into a 3-qubit state that corrects against Z errors. (Your circuit should use standard gates like H, $CNOT$, etc.)

   (b) Show how to detect Z errors using two ancillary qubits, control operations, and a measurement of the ancillary qubits.

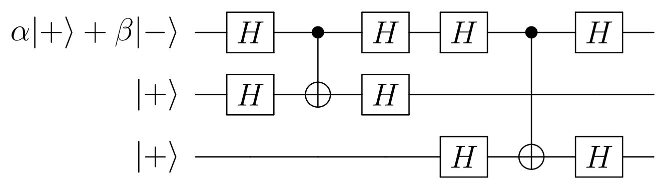
   (c) What is the projective measurement that your circuit in part (b) accomplishes?

Solution

(a) We first write the initial state in the $|+\rangle$, $|-\rangle$, basis:

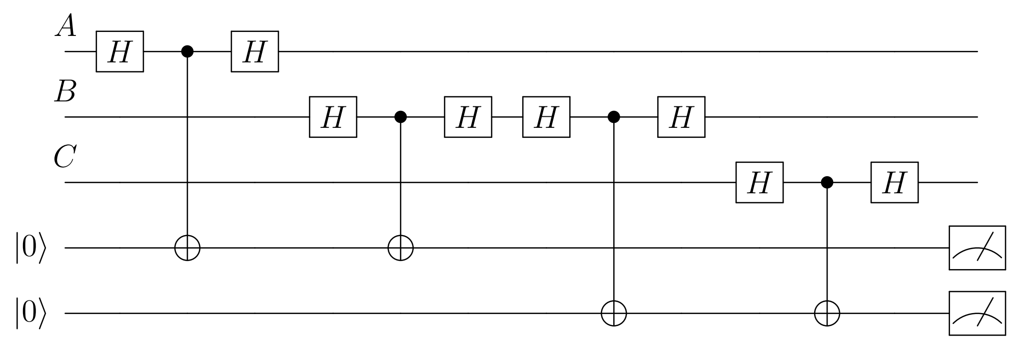
$$\alpha|+\rangle + \beta|-\rangle, \tag{11}$$

where $\alpha = \frac{1}{\sqrt{2}}(a+b)$ and $\alpha = \frac{1}{\sqrt{2}}(a-b)$. Now we would like to create the state $\alpha|+++\rangle + \beta|---\rangle$ because then if a Z error occurs on one qubit, we can do a majority vote among $+$ and $-$ to detect and correct. To do this, we want to append 2 qubits in the state $|+\rangle$, and then if our initial qubit is in the state $|-\rangle$, we want to flip these qubits to also be in the state $|-\rangle$. In other words, we want to apply the operation $|+\rangle\langle+| \otimes I + |-\rangle\langle-| \otimes Z$. This is the same as $(H \otimes H)CNOT(H \otimes H)$. Thus our full circuit is



(12)

(b) We again attach two ancillary qubits, but instead of wanting to flip the target when the control has value $|1\rangle$, we want to flip when the control has value $|-\rangle$, so we apply H only on the control qubits around the CNOT. This results in the following circuit (the input is $\alpha|+++\rangle_{ABC} + \beta|---\rangle_{ABC}$).



(13)

(c) The projectors of our effective measurement are

$$\begin{aligned}
P_0 &= |+++\rangle\langle+++| + |---\rangle\langle---| \\
P_1 &= |-++\rangle\langle-++| + |+--\rangle\langle+--| \\
P_2 &= |+-+\rangle\langle+-+| + |-+-\rangle\langle-+-| \\
P_3 &= |++-\rangle\langle++-| + |--+\rangle\langle--+|
\end{aligned} \tag{14}$$

Hints on 1:

- $(a \oplus b) \cdot c \equiv a \cdot c + b \cdot c \mod 2$.
- There should be two standard basis states in superposition after the measurement of the second register.

Hints on 2:

- Convert the initial state to $|+\rangle/|-\rangle$ basis. Then think about how you can convert the $|+\rangle$ into $|+++\rangle$ and the $|-\rangle$ into $|---\rangle$ similarly to what we do in the bit-flip case.