CS333 - Qubit Worksheet

1. If you measure the state $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + i\sqrt{\frac{2}{3}}|1\rangle$ in the standard basis, what happens, and with what probability? What about if you measure using the basis $\{|\rightarrow\rangle, |\leftarrow\rangle\}$?

   Solution With the first measurement, you get outcome $|0\rangle$ with probability 1/3 and outcome $|1\rangle$ with probability 2/3. With the second measurement, you get outcome $|\rightarrow\rangle$ with probability $\frac{(\sqrt{2}+1)^2}{6}$, and outcome $|\leftarrow\rangle$ with probability $\frac{(\sqrt{2}-1)^2}{6}$.

   See next page for more detailed notes on how to get these solutions, using both vector and ket notation.

2. If you have two qubits states $|\psi\rangle$ and $|\phi\rangle$ such that $\langle\psi|\phi\rangle = 0$, explain what measurement you should use to perfectly distinguish between these two states?

   Solution Note that $\{|\psi\rangle, |\phi\rangle\}$ is an orthonormal basis, since $\langle\psi|\psi\rangle = \langle\phi|\phi\rangle = 1$ because $|\phi\rangle$ and $|\psi\rangle$ represent qubit states, and $\langle\phi|\psi\rangle = 0$. So $\{|\psi\rangle, |\phi\rangle\}$ represents a valid quantum meausurement. If we use this measurement to measure either $|\phi\rangle$ or $|\psi\rangle$, we will get each outcome with probability 1 or 0, so we perfectly distinguish between these states.

3. Explain what will happen in the following situations using bra and ket notation, and the language of collapse:

   (a) Alice prepares a right diagonal photon, and Bob measures it using a vertically polarized filter.

   (b) Alice prepares a left diagonal photon, and Bob measures it using a right diagonally polarized filter.

   Solution

   (a) Alice prepares the state $|+\rangle$, and Bob measures in the $\{|0\rangle, |1\rangle\}$ basis. Therefore, he gets outcome $|0\rangle$ with probability $|\langle 0|+\rangle|^2 = 1/2$. In this case, the qubit collapses to the state $|0\rangle$ and so passes through the filter. He gets outcome $|1\rangle$ with probability $|\langle 1|+\rangle|^2 = 1/2$. In this case, the state collapses to the state $|1\rangle$ and gets blocked by the filter.

   (b) Alice prepares the state $|-\rangle$, and Bob measures in the $\{|+\rangle, |-\rangle\}$ basis. Therefore, he gets outcome $|-\rangle$ with probability $|\langle -|-\rangle|^2 = 1$. In this case, the qubit doesn't collapse (it stays in the same state), which gets blocked by the filter, so nothing emerges.

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1st case: $M = \{|0\rangle, |1\rangle\}$

- Outcome $|0\rangle$

$$|\langle 0|\psi\rangle|^2 = \left|\langle 0|\left(\frac{1}{\sqrt{3}}|0\rangle + i\sqrt{\frac{2}{3}}|1\rangle\right)\right|^2$$

$$= \Big|\frac{1}{\sqrt{3}}\underbrace{\langle 0|0\rangle}_{=1} + i\sqrt{\frac{2}{3}}\underbrace{\langle 0|1\rangle}_{=0}\Big|^2$$

$$= \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3} \quad \text{probability}$$

or $$|\langle 0|\psi\rangle|^2 = \left|\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ i\sqrt{\frac{2}{3}} \end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$$

- Outcome $|1\rangle$

$$|\langle 1|\psi\rangle|^2 = \left|\langle 1|\left(\frac{1}{\sqrt{3}}|0\rangle + i\sqrt{\frac{2}{3}}|1\rangle\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{3}}\langle 1|0\rangle + i\sqrt{\frac{2}{3}}\langle 1|1\rangle\right|^2$$

$$= \left|i\sqrt{\frac{3}{3}}\right|^2 = i\sqrt{\frac{2}{3}}\cdot(-i)\cdot\sqrt{\frac{2}{3}} = \frac{2}{3} \quad \text{probability}$$

or $$|\langle 1|\psi\rangle|^2 = \left|\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ i\sqrt{\frac{2}{3}} \end{pmatrix}\right|^2 = \left|i\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3}$$

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2^{nd} Case $\quad M = \{|\rightarrow\rangle, |\leftarrow\rangle\}$

- Outcome $|\rightarrow\rangle$

-i b/c of conjugation

$$|\langle\rightarrow|\psi\rangle|^2 = \left|\left(\frac{1}{\sqrt{2}}\langle 0| - \frac{i}{\sqrt{2}}\langle 1|\right)\left(\frac{1}{\sqrt{3}}|0\rangle + i\sqrt{\frac{2}{3}}|1\rangle\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{6}}\langle 0|0\rangle - \frac{i}{\sqrt{6}}\langle 1|0\rangle + i\sqrt{\frac{1}{3}}\langle 0|1\rangle + \frac{1}{\sqrt{3}}\langle 1|1\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}\right|^2 = \left|\frac{1+\sqrt{2}}{\sqrt{6}}\right|^2 = \frac{(1+\sqrt{2})^2}{6}$$

or

$$\left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1/\sqrt{3} \\ i\sqrt{\frac{2}{3}}\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}} + \sqrt{\frac{2}{\sqrt{3}}}\right)\right|^2 = \frac{(1+\sqrt{2})^2}{6}$$

- Outcome $|\leftarrow\rangle$

$$|\langle\leftarrow|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}} \\ i\sqrt{\frac{2}{3}}\end{pmatrix}\right| = \left|\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}} - \frac{\sqrt{2}}{3}\right)\right|^2$$

$$= \frac{(\sqrt{2}-1)^2}{6}$$