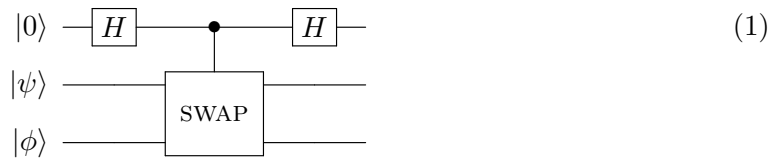# CS333 - Problem Set 9

See final page for hints.

1. (a) Consider the following circuit on 3 qubits. Let $|\psi\rangle$ and $|\phi\rangle$ be any single-qubit states (not necessarily standard basis states), and let SWAP denote the 2-qubit gate that swaps its input qubits (i.e., SWAP$|\eta\rangle|\mu\rangle = |\mu\rangle|\eta\rangle$ for any states $|\mu\rangle$ and $|\eta\rangle$). What is the final state of the following circuit (in terms of $|\psi\rangle$ and $|\phi\rangle$ )?

$$
\begin{array}{c}
|0\rangle \quad \boxed{H} \quad \bullet \quad \boxed{H} \\
|\psi\rangle \quad \\
\quad \boxed{\text{SWAP}} \\
|\phi\rangle \quad
\end{array}
\tag{1}
$$

(b) Suppose the first (top) qubit in the above circuit is measured in the standard basis. What is the probability that the measurement outcome is $|0\rangle$? Your answer should depend on the inner product of $|\psi\rangle$ and $|\phi\rangle$.

(c) How do the results of the previous parts change if $|\psi\rangle$ and $|\phi\rangle$ are $n$-qubit states, and SWAP denotes the $2n$-qubit gate that swaps the first $n$ qubits with the last $n$ qubits?

(d) What is the purpose of this circuit?

2. Analysis of details of period finding algorithm we skipped in class :)

(a) In class, we showed that the probability of getting an outcome $|y\rangle$ in the final measurement of the period finding algorithm is

$$
Pr(|y\rangle) = \frac{1}{Nm_{b^*}} \left| \sum_{m=0}^{m_{b^*}-1} e^{-2\pi imry/N} \right|^2 .
\tag{2}
$$

Using the geometric series formula, and the fact that

$$
\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i},
\tag{3}
$$

show that when $y$ is not a multiple of $N/r$,

$$
Pr(|y\rangle) = \frac{\sin^2(\pi rym_{b^*}/N)}{Nm_{b^*}\sin^2(\pi ry/N)}.
\tag{4}
$$

and that when $y$ is a multiple of $N/r$,

$$
Pr(|y\rangle) = m_b^*/N.
\tag{5}
$$

(b) Plot $Pr(|y\rangle)$ (note this function is not continuous!) for $N = 400$, $r = 7$, and $m_b^* = \lceil N/r \rceil$ using any plotting software (e.g. matplotlib for python, mathematica, etc). (In this case, is $y$ ever a multiple of $N/r$?)

(c) You should find that there are several values of $y$ that are particularly likely. Choose one of these - call it $y^*$. Find the continued fraction of $y^*/N = y^*/400$. (See [Wikipedia on Continued Fractions](#)). The convergents of a continued fraction are when you stop the algorithm at an intermediate step. Using the example on the wikipedia page, the first convergent is 3, the second is $3 + \frac{1}{4}$, the third is $3 + \frac{1}{4+\frac{1}{12}}$, and the fourth is $3 + \frac{1}{4+\frac{1}{12+\frac{1}{4}}}$.

Then our guess for $r$ (or at least a factor of $r$) is the denominator of the convergent (after converting the convergent into a simple fraction) that is closest to $y^*/N$ while still having denominator less than $\sqrt{N}$. Show this approach works, at least for this example. It turns out you can prove this works in general with high probability.

3. In the standard quantum search algorithm, there is exactly one "marked" input to $f$. (In other words there is one input $s$ where $f(s) = 1$). Now suppose that there are two inputs $s$ and $t$ such that $f(s) = f(t) = 1$, and the rest of the inputs to $f$ have value 0. So there are two "marked" inputs, and our goal is to find either $s$ or $t$.

We consider running exactly the same algorithm as in standard search.

(a) Explain why $U_f = \mathbb{I} - 2|s\rangle\langle s| - 2|t\rangle\langle t|$.

(b) In search with one marked item, we considered a 2-D axes with $y$-axis $|s\rangle$ and $x$-axis $|\beta\rangle = \frac{1}{\sqrt{N-1}}\sum_{i\neq s}|i\rangle$. Now we consider $y$-axis $\frac{1}{\sqrt{2}}(|s\rangle + |t\rangle)$ and $x$-axis $|\beta'\rangle = \frac{1}{\sqrt{N-2}}\sum_{i\neq s,t}|i\rangle$. If we plot the equal superposition state as a point on these axes, what is the angle between the equal superposition state and the $x$-axis?

(c) Now the algorithm proceeds as before with the same two reflections. What is the runtime of the algorithm? How does this compare to the standard Grover search algorithm?

# Hints!

1. (a) Hint: don't expand $|\psi\rangle$ using standard basis states. That is, don't write $|\psi\rangle = a|0\rangle + b|1\rangle$. Instead, just keep as $|\psi\rangle$.

   Solution: $\frac{1}{2}[|0\rangle(|\psi\rangle|\phi\rangle + |\phi\rangle|\psi\rangle) + |1\rangle(|\psi\rangle|\phi\rangle - |\phi\rangle|\psi\rangle)]$

   (b) Solution: $\frac{1}{4}\left(2 + 2\,|\langle\psi|\phi\rangle|^2\right)$. Remember a state is normalized if it's inner product with itself is 1.