CS333 - Problem Set 5

1. Consider the entangled state

$$|\phi\rangle = \frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle. \tag{1}$$

   (a) Suppose the first qubit of $|\phi\rangle$ is measured in the standard basis, i.e. $\{|0\rangle, |1\rangle\}$. What is the probability of obtaining outcome $|0\rangle$, and in the event that this outcome occurs, what is the resulting state of the second qubit?

   (b) (Assume you have a clean copy of $|\phi\rangle$ with no measurement applied for this problem.) Suppose the second qubit of $|\phi\rangle$ is measured in the basis $\{|+\rangle, |-\rangle\}$. What is the probability of obtaining $|+\rangle$, and in the event that this outcome occurs, what is the resulting state of the first qubit?

   (c) (Assume you have a clean copy of $|\phi\rangle$ with no measurement applied for this problem.) Suppose the first qubit of $|\phi\rangle$ is measured in the standard basis and the second qubit is measured in the $\{|+\rangle, |-\rangle\}$ basis. What are the probabilities of the four possible outcomes? Show that they are consistent with the marginal probabilities you computed in the previous two parts.

   (d) We know entangled states have weird properties, so perhaps if Alice measures first, that will affect Bob's probability distribution, or vice versa. What does this problem imply regarding the order in which Alice and Bob measure there qubits? What does this mean in terms of the possibility of faster than light communication?

   (e) (Assume you have a clean copy of $|\phi\rangle$ with no measurement applied for this problem.) Suppose the H gate is applied to the second qubit of $|\phi\rangle$, and then the second qubit is measured the standard basis.

      i. What is the state after H is applied?

      ii. What is the probability of obtaining outcome $|0\rangle$, and in the event that this outcome occurs, what is the resulting state of the second qubit?

      iii. Can you guess from this example the connection between unitaries and measuring in different bases?

2. **[This problem moved to the next problem set]** Alice and Bob would like to share the entangled state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Unfortunately, they do not initially share any entanglement. But fortunately, they have a mutual friend, Charlie, who shares a copy of $|\beta_{00}\rangle$ with Alice and another copy of $|\beta_{00}\rangle$ with Bob.

   (a) Write the initial state using ket notation as a superposition of standard basis states, and use subscripts to indicate which qubit is in which person's posession. The first qubit should belong to Alice, the second and third qubits belong to Charlie (the second is entangled with Alice's qubit and the third is entangled with Bob's qubit), and the fourth qubit belongs to Bob.

(b) Suppose Charlie performs a Bell measurement on his two qubits (one of which is entangled with Alice and the other of which is entangled with Bob). For each possible measurement outcome, give the probability with which it occurs and the resulting post-measurement state for Alice and Bob. Note the Bell measurement uses the basis:

$$\begin{aligned} |\beta_{00}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ |\beta_{01}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |\beta_{10}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |\beta_{11}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{aligned} \tag{2}$$

(c) Consider the system after Charlie's measurement, and design a protocol whereby Charlie sends a classical message to Alice, and Alice applies a unitary to her quantum state based on that message, such that after doing this, Alice and Bob share the state $|\beta_{00}\rangle$.

3. Eve knows it is impossible to create a generic cloner, but she thinks she has found a pretty good cloner that will help her break the BB84 cryptography scheme. For each round of the cryptography protocol, she prepares a qubit in the state $|0\rangle_E$. When the photon that Alice is sending to Bob comes to her, she applies the unitary:

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3}$$

to the combined state of $|\psi\rangle|0\rangle_{AE}$ (where $|\psi\rangle_A$ is the state of Alice's photon.) Then after acting with the unitary, she sends qubit A to Bob, and keeps system E but doesn't measure it yet. When Alice and Bob announce their measurement bases, if Alice and Bob's measurement bases were the same, Eve measures her system in that basis.

(a) In what way does $CNOT$ act like a cloner? When does $CNOT$ act like a cloner and when does it not? Please explain.

(b) Is this a good strategy? Compare to Eve's previous strategy where she always chooses to measure each incoming photon in the basis $\{|0\rangle, |1\rangle\}$.