CS333 - Problem Set 3

1. Consider the entangled 2-qubit state $|\psi\rangle = \frac{1}{\sqrt{2}}|01\rangle_{AB} - \frac{1}{\sqrt{2}}|10\rangle_{AB}$. This state has some strange properties - in particular the two qubits are perfectly anticorrelated. It's behavior is so strange, it caused Einstein to believe that quantum mechanics couldn't possibly describe reality.

   (a) Suppose Alice and Bob each measure their qubit in the same basis. That is, Alice and Bob both each apply the measurement $M(\eta, \chi) = \{|\phi_0(\eta, \chi)\rangle, |\phi_1(\eta, \chi)\rangle\}$, where $|\phi_0(\eta, \chi)\rangle = \cos\eta|0\rangle + e^{i\chi}\sin\eta|1\rangle$ and $|\phi_1(\eta, \chi)\rangle = -\sin\eta|0\rangle + e^{i\chi}\cos\eta|1\rangle$. Show that if Alice gets outcome $|\phi_0(\eta, \chi)\rangle$, Bob will get outcome $|\phi_1(\eta, \chi)\rangle$, or vice versa. $M(\eta, \chi)$ is "generic" in that by choosing any various real numbers for χ and η, we can make $M(\eta, \chi)$ represent any possible single qubit measurement. Please calculate the probability of at least one outcome by hand for practice. If you feel like you don't need additional practice, you may use a computer to calculate the others. (η is "eta," χ is "chi.")

   (b) What is the probability that Alice gets outcome $|\phi_0(\eta, \chi)\rangle$? What is the probability that she gets $|\phi_1(\eta, \chi)\rangle$?

   (c) This anticorrelation is strange because of the following thought experiment. Suppose Alice took her qubit to the moon, and Bob stayed on Earth. Now Alice performs her measurement first, and suppose she gets outcome $|\phi_0(\eta, \chi)\rangle$. Then, even if Bob performs his measurement before any lightspeed communication can have happened between Alice and Bob, Bob's qubit somehow knows to choose the opposite outcome. This will happen no matter which values of η and χ Alice and Bob choose. People thought that this might enable faster-than-light communication, but explain why Part (b) of this question rules out faster-than-light communication.

   (d) We also can have classical (non-quantum) bits that are probabilistically anti-correlated. Consider the following situation. I put a red sock in one box and a blue sock in another box, and I give one box to Alice and one box to Bob, without telling them which box contains which sock. Once Alice opens her box and sees a blue sock, she immediately knows that Bob's box contains a red sock. Why is the quantum situation stranger than this classical situation?

2. Alice and Bob are playing a game where the referee sends them each a qubit that is part of a 2-qubit state. The referee promises that the 2-qubit state is one of two options: $|\psi_0\rangle$ or $|\psi_1\rangle$, but doesn't tell them which. Alice and Bob's goal is to decide which state they were given. In order to do this, they can each make a local measurement on their part of the state (i.e. they can make a quantum measurement on their individual qubit). After the measurement, they can communicate classically (for example, have a conversation on the phone), discuss their outcomes, and try to decide if the state is $|\psi_0\rangle$ or $|\psi_1\rangle$. For each of the following games, either give a strategy such that Alice and Bob can always win, or explain why they will always lose with some probability.

(a)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB}) \tag{1}$$

(b)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$$

$$|\psi_1\rangle = |00\rangle_{AB} \tag{2}$$

(c)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{AB} - |11\rangle_{AB}) \tag{3}$$

3. Not all entangled states are equal. Some states are more entangled than others. In this problem, you will investigate how the level of entanglement in Alice and Bob's shared state affects their ability to win the CHSH game. (Warning - this problem is calculation heavy, but good practice!)

   (a) Let $|\psi(\theta)\rangle$ be the two-qubit state $|\psi(\theta)\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$. For what values of θ is $|\psi(\theta)\rangle$ entangled? (Please only consider $0 \leq \theta \leq \pi/2$.)

   (b) Suppose Alice and Bob share a state $|\psi(\theta)\rangle$ as a resource, and use the same strategy we discussed in class to try to win the CHSH game. What is Alice and Bob's average probability of winning the game, as a function of θ, where the average is taking over the possible values of x and y? (You can halve the number of calculations you have to do by just calculating the probability of the winning outcomes.) (Again, please calculate the probability of at least one outcome by hand for practice. If you feel like you don't need additional practice, you may use a computer to calculate the others.)

   (c) Do Alice and Bob always do better than the best classical strategy, as long as they have some entanglement? Or do they need a certain amount of entanglement (certain values of θ)?

4. The problem with interferometers is moved to next week.