S. KIMMEL

Goals

Path where each vertex is hit once exactly

- Show HAMPATH is NP Hard
- Show Subset Sum is NP Hard

will explain

Review Requests @ CANVAS

Strategy:

Show polynomial reduction from 3SAT to new problem

3SAT: Given CNF formula of $x_1, x_2, \ldots x_n$ and negations

AND of ORs

with m clauses where each clause has at most 3 terms, is there a satisfying assignment?

Example input:

$$\underbrace{(x_1 \lor x_3 \lor \bar{x}_4)}_{C_1} \land \underbrace{(\bar{x}_1 \lor \bar{x}_2 \lor \bar{x}_5)}_{C_2} \land \underbrace{(\bar{x}_3 \lor x_5)}_{C_3}$$

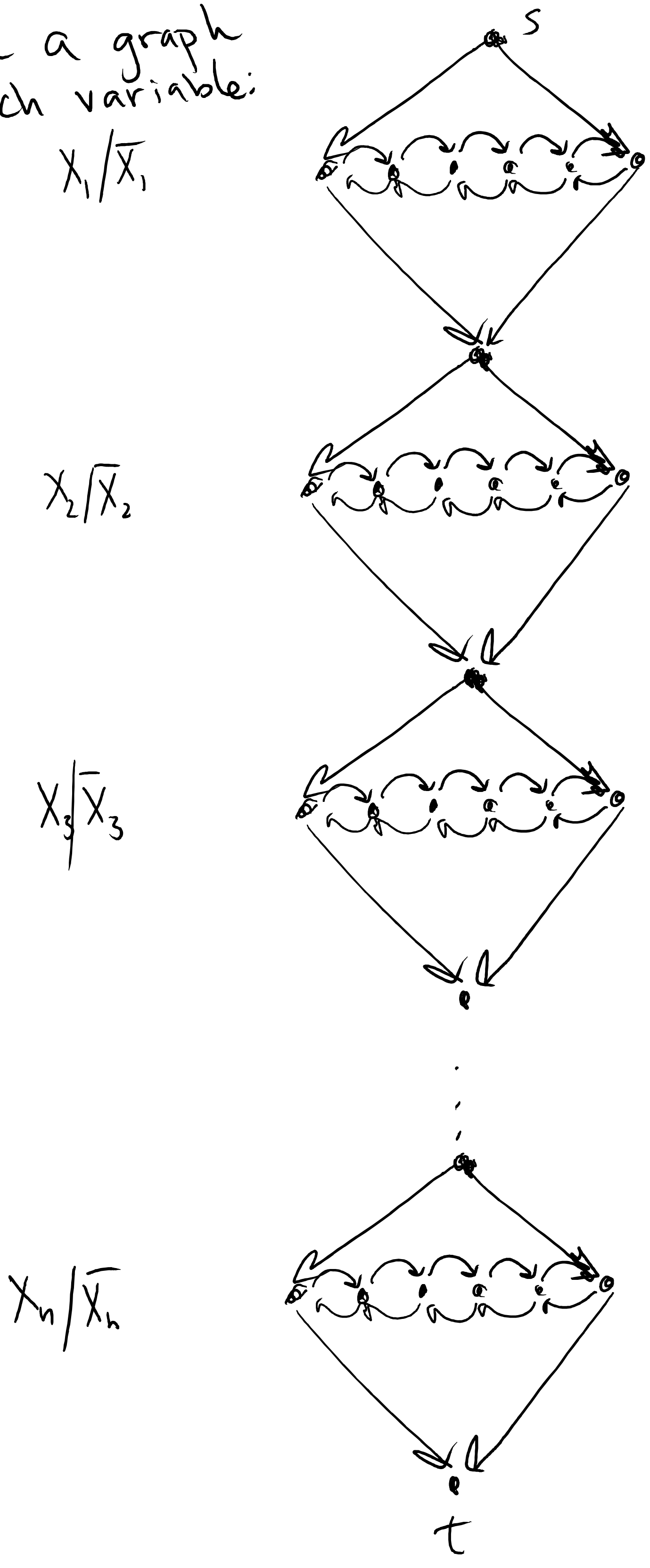
Clauses: C_1 C_2 C_3

Reduction

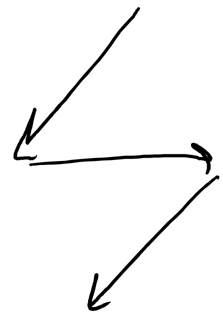
3SAT x → [convert] → HAMPATH x' → [HAMPATH SOLVER] → Yes / No → [convert] → 3SAT x = yes or no

S.KIMMEL

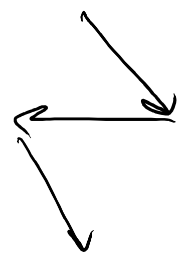
Create a graph for each variable:



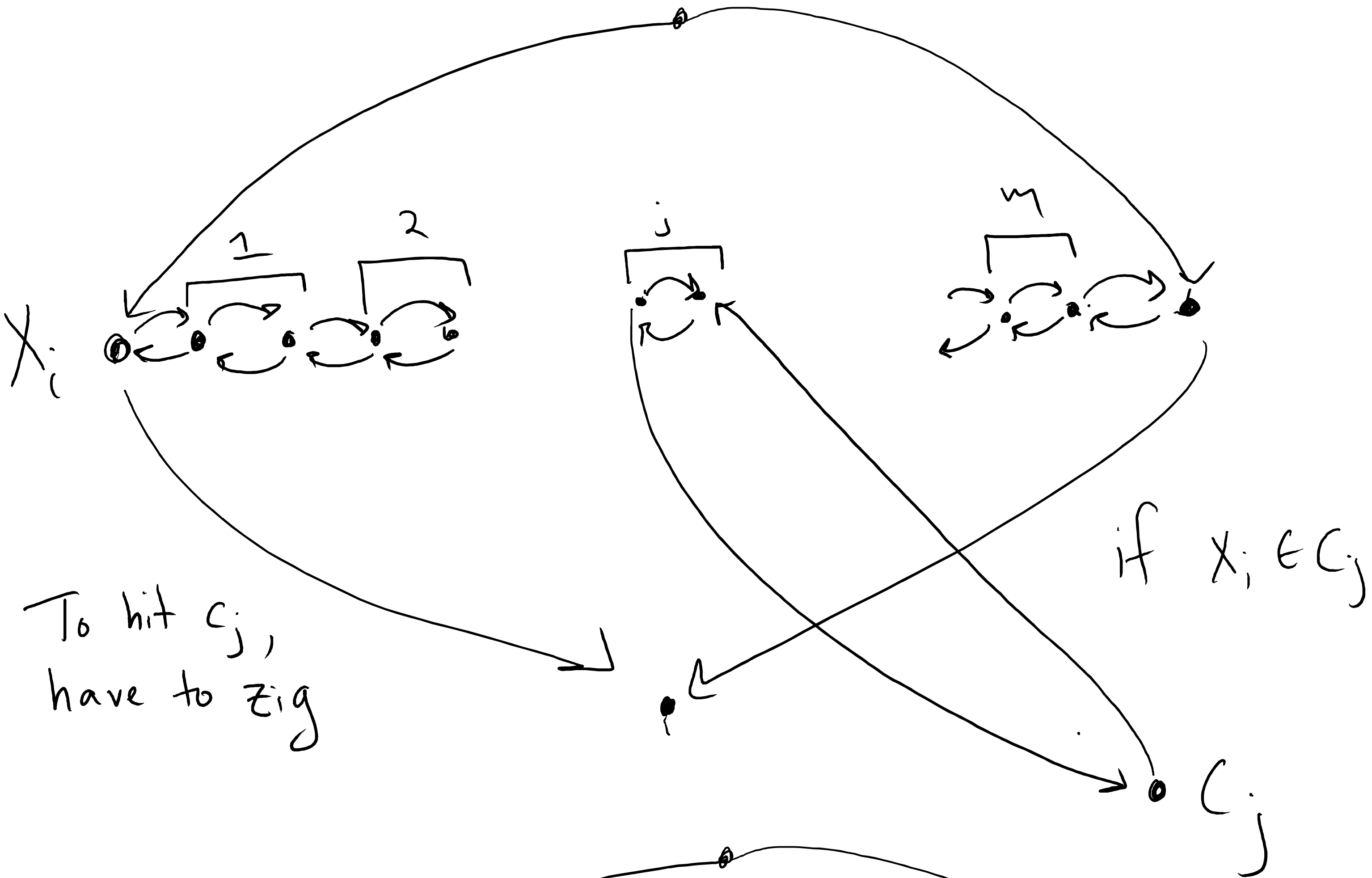
If $X_i = 1$ to satisfy 3SAT, will force path to zig



If $X_i = 0$, will force zag:

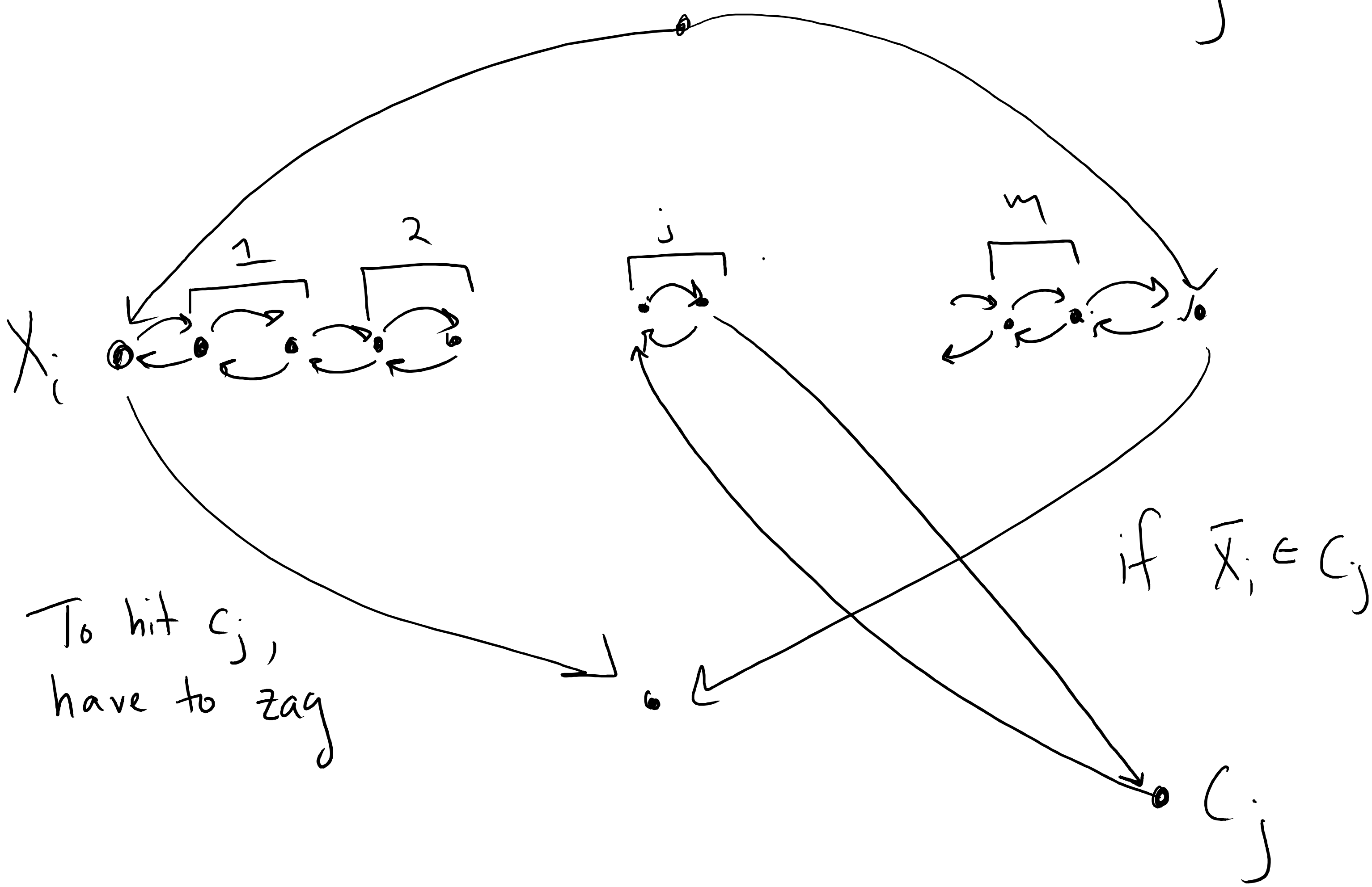


S. KIMMEL



if $x_i \in C_j$

To hit c_j, have to zig



if $\bar{x}_i \in C_j$

To hit c_j, have to zag

S.KIMMEL

$(x_1 \vee \bar{x}_2 \vee x_3)$

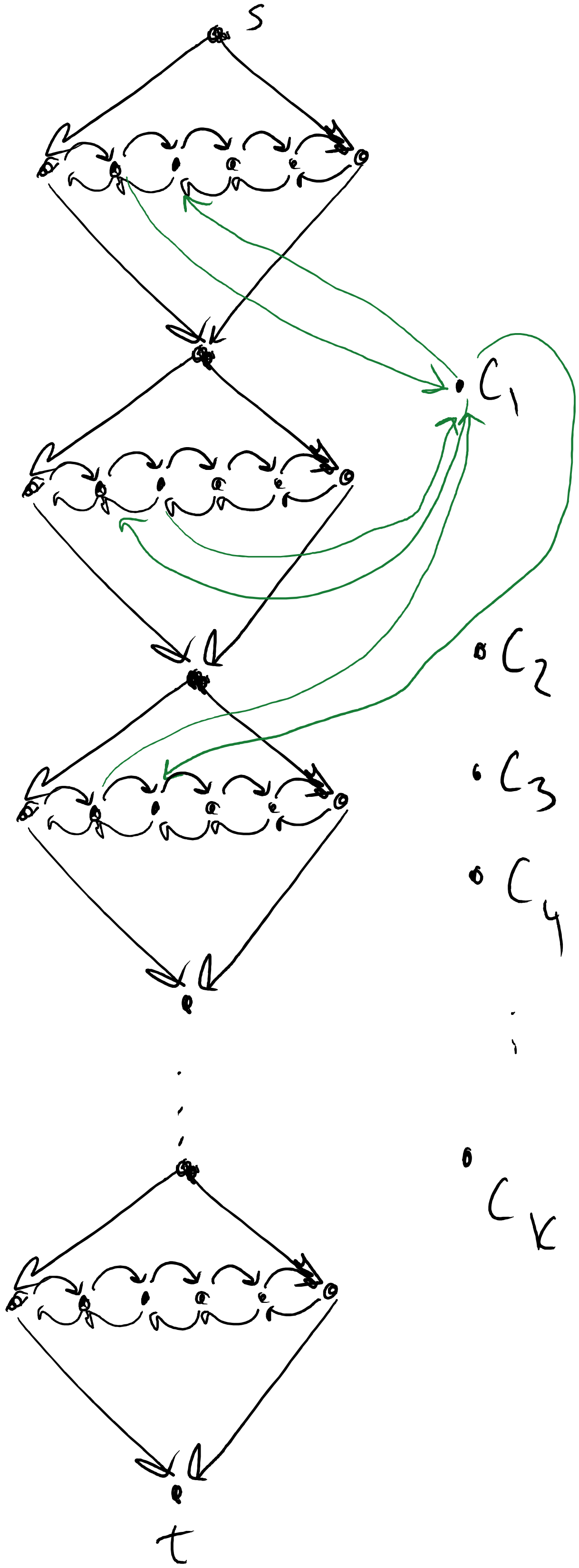
Draw

$x_1 / \bar{x}_1$

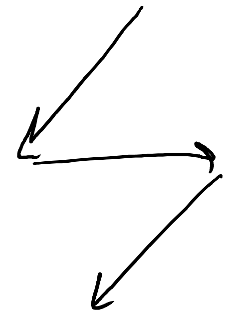
$x_2 / \bar{x}_2$

$x_3 / \bar{x}_3$

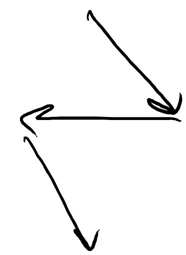
$x_n / \bar{x}_n$



$X_i = 1$: zig



$X_i = 0$: zag



Must hit vertex C_1. Can get to it from one of the 3 variables in clause. Forces zig/zag for that variable.

S. KIMMEL

Show Polynomial Reduction:

1: Poly time to create graph from 3SAT input.

2. Poly time to convert output of HAMPATH to output of 3SAT

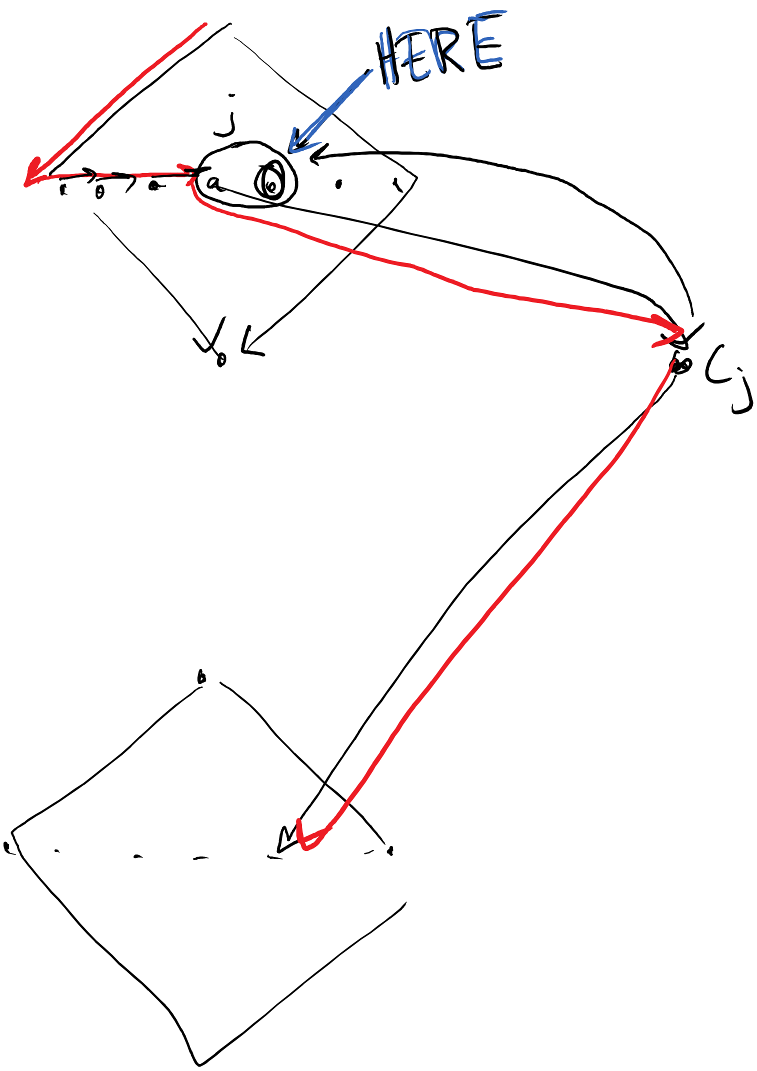
- If 3SAT is satisfiable then HAMPATH exists

  Pf: If 3SAT soln is $X_i = 1$, zig
  If 3SAT soln is $X_i = 0$ zag
  Each clause vertex must be going in same direction as zig or zag for at least one of its three elements. Use one of those elements to visit the clause vertex.

- If 3SAT is not satisfiable, HAMPATH does not exist.

  Pf: If HAMPATH exists and zigs/zags through each diamond, assign $X_i = 1$ or 0 according to zig or zag. Otherwise, the path can use clause to leave a diamond and enter another

S.KIMMEL



But then can't get back to HERE without hitting c_j again, so not a HAMPATH

S. KIMMEL

Subset Sum

Input: $S = \{w_1, \ldots, w_k\}, \ t$

Output: $\{y_1, \ldots, y_\ell\} \subseteq S : \ \sum_{i=1}^{\ell} y_i = t$

ex: $S = 1, 2, 2, 6, 7, 10$

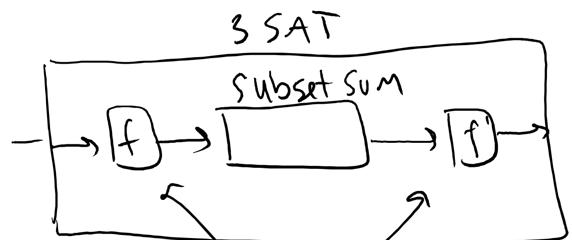
$t = 15$:

$6, 7, 2$

Goal: Show Subset-Sum is NP-Hard

Strategy: Prove 3-SAT reduces to subset sum

1.



Create f, f'

2. Show f, f' take Polynomial time

3. S-S has soln $\Longleftrightarrow$ 3-SAT has solution

3-SAT is NP Hard, so if we show 3-SAT reduces to S-S, this means S-S is even harder than 3-SAT

3-SAT

Input: $x_1, \ldots x_n$, variables, $c_1, \ldots, c_m$ clauses each involving ≤ 3 variables

Output: assigment that satisfies all clauses

Clause: $(x_1 \vee \bar{x}_2 \vee \bar{x}_3)$

S.KIMMEL

#'s in set: $\overbrace{n+m \text{ digits}}$ Not Binary

	1	2	3	…	n	c_1	c_2	…	c_m
$x_1 \to w_1 =$	1	0	0	0.000	0	1	0	….	0
$\bar{x}_1 \to w_2 =$	1	0	0	0000	0	0	1	….	0
$x_2 \to w_3 =$		1	0	00.00	0	0	0	…	0
$\bar{x}_2 \to w_4 =$		1	0	0.0.00	0	1	0	….	0
$x_3 \to w_5 =$			1	0000	0	0	0	…	0
$\bar{x}_3 \to w_7 =$			1	0000	0	1	1	…	0
⋮				⋮				⋮	
$x_n \to w_{2n-1} =$					1	0	0	…	1
$\bar{x}_n \to w_{2n} =$					1	0	0	…	0
$w_{2n+1} =$						1	0	0 0000	0
$w_{2n+2} =$						1	0	0 000	0
$w_{2n+3} =$							1	0 000	0
$w_{2n+4} =$							1	0 000	0
⋮								⋮	
$w_{2n+2m-1}$									1
w_{2n+2m}									1
t	1	1	1	…	1	3	3	…	3

w_{2n+1} through w_{2n+2m}: 2m numbers

$c_1 = (x_1 \vee \bar{x}_2 \vee \bar{x}_3)$

$c_2 = (\bar{x}_1 \vee \bar{x}_3)$

⋮

$c_m = (x_n \vee \dots)$

$$W_1 = \underbrace{100,000,000}_{n},\underbrace{100,000}_{m}$$

← Not binary!

100 trillion, 100 thousand

S.KIMMEL

1. 3SAT → Subset sum conversion in P:
   - Poly # of #'s, each with Poly digits)

2. Poly time to convert output of SUBSET SUM to output of 3SAT

- If 3-SAT has a solution, then subset sum has solution

  → Include $\begin{cases} w_{2i-1} \text{ \# if } x_i = 1 \\ w_{2i} \text{ \# if } \bar{x}_i = 0 \end{cases}$

  Then every clause digit has at least 1 (max 3) add 2 more from w_{2n+2j}, $w_{2n+2j+1}$ to get to 3.

- If 3-SAT has no solution, then subset sum has no solution.

  We prove contrapositive

  - Note: no carry over, so if get to t,
    - → one of x_i, $\bar{x}_i$ is in subset
    - → without w_{2n+} terms, each digit $c_1, \ldots c_m$ has at least 1 contributing → each clause has at least one satisfying variable