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Groups

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Reflection: Use group outside class. Look up master metho?

Programming Assignment!

Goals

- Create + analyze loop invariants
- Analyze QuickSort

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Loop Invariants: Prove loops are correct

```
setup
while (condition) {
      stuff
}
Great output
```

Induction tailored to loops

Parts of Loop Invariant Proof

1. State Invariant: thing(s) that is true before & after each loop iteration

2. Base Case: Show invariant is true before loop starts.

3. Maintenance: Show if invariant is true before an iteration, it is true after an iteration

4. Termination: argue loop ends. Given status of invariant after final loop, argue great output.

Input : Array A of integers of length n

Output: Array containing sorted elements of A

for $k = 1$ **to** $n - 1$ **do**
    for $j = n$ **to** $k + 1$ **do**
        if $A[j] < A[j - 1]$ **then**
            Swap $A[j]$ and $A[j - 1]$;
        end
    end
end
return A;

Algorithm 2: $\texttt{BubbleSort}(A)$

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Bubble Sort:

1. Inner Loop Invariant:

   - $A[j]$ is the smallest element of $A[j:n]$
   - The elements of A are same as input array.

   Base case: $j=n$, $A[n]$ is smallest of $A[n:n]$

   Maintenance: Since $A[j]$ is smallest of $A[j:n]$, it is smaller than all in $A[j-1:n]$ except perhaps $A[j-1]$, but we compare $A[j-1]$ and $A[j]$ and swap smaller to $A[j-1]$. This preserves elements of A and ensures after loop, $A[j-1]$ is smallest element of $A[j-1:n]$.

   Termination: The loop terminates at $j=k$, so we have
   - $A[k]$ is smallest element of $A[k:n]$
   - Elements of A preserved.

2. Outer loop invariant:

   - $A[1:k-1]$ is sorted
   - $A[1:k-1]$ contains the smallest $k-1$ elements of array
   - Elements of A same as input

   Base case: $k=0$, no elements in $A[1:k]$, A same as input

   Maintenance: Start of loop, $A[1:k-1]$ is sorted + contains $k-1$ smallest elements of A. Then inner loop moves smallest of remaining $A[k:n]$ to $A[k]$ while preserving elements, so now $A[1:k]$ contains smallest k elements of A, sorted.

   Termination: At $k=n$, so $A[1:n-1]$ is sorted smallest elements, but there is only one remaining element in $A[n]$, which must be the largest element. Elements are same as input, so output is sorted array.

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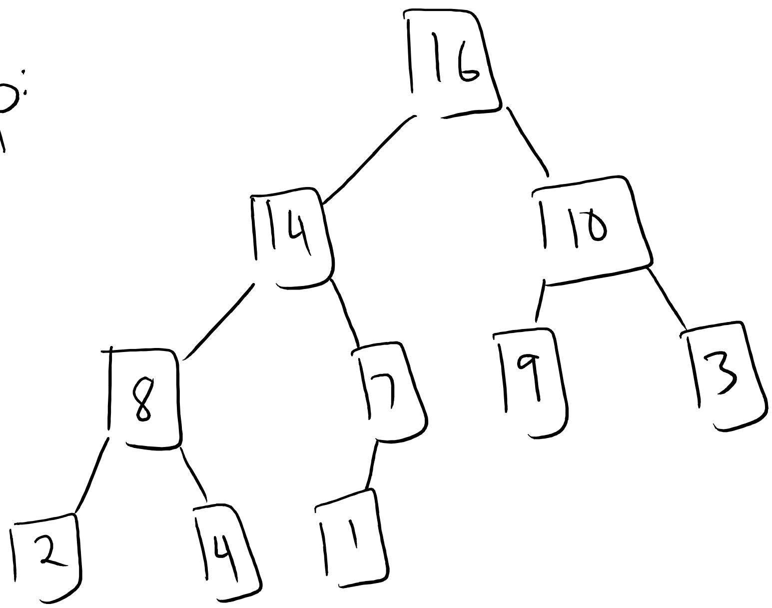
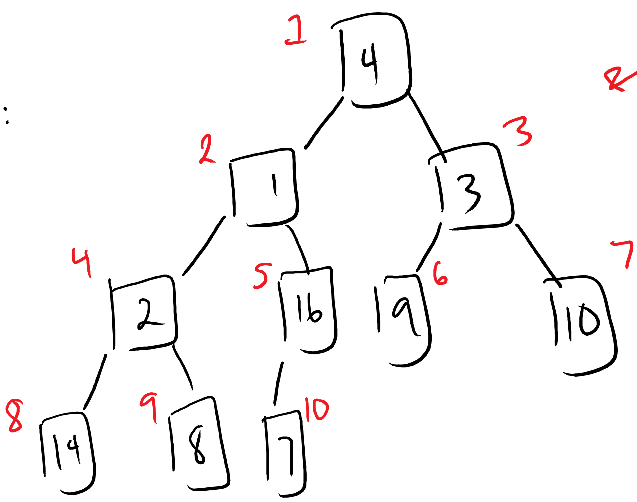
Loop Invariant for Heapify

Max Heap:



The key of each node is larger than all of its descendants

Before creating a heap:



In red: indeces of array where heap is stored:

1	2	3	4	5	...
4	1	3	2	16	...

Build-Max-Heap

for i = $\lfloor A.length/2 \rfloor$ to 1

    Max-Heapify(A, i)

All indeces > $\lfloor A.length/2 \rfloor$ are leaves

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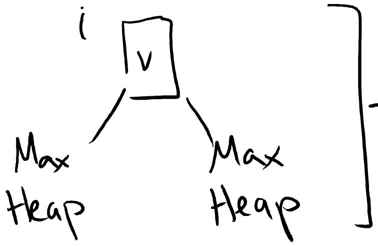
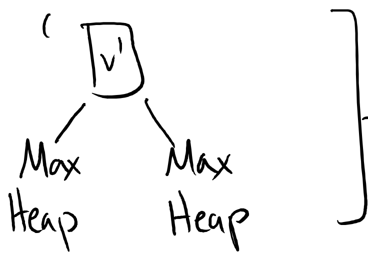
Max Heapify(A, i)

Input:



→ Full tree NOT max heap

Output:



→ Max heap

Prove Build-Max-Heap works correctly:

Invariant:

Initialization

Maintenance

Termination

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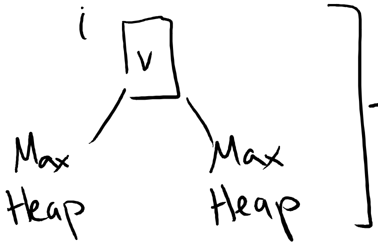
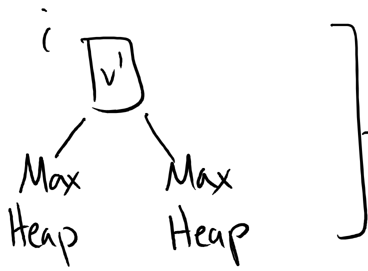
Max Heapify(A, i)

Input :



→ Full tree NOT max heap

Output:



→ Max heap

Prove Build-Max-Heap works correctly:

Invariant: Indeces $i+1, i+2, \ldots, n$ are roots of max heaps

Initialization

Indeces $\lfloor A.length/2 \rfloor, \ldots, n$ are leaves, and trees with one node are max heaps.

Maintenance

By our invariant, the children of i are roots of heaps, so Max-Heapify creates heap at i. Now $i, i+1, \cdots, n$ are roots of heaps.

Termination

Loop ends at $i=1$, so indeces $1, 2, \cdots, n$ are roots of heaps but in particular, 1 is a root of a heap, so the whole tree is a heap.

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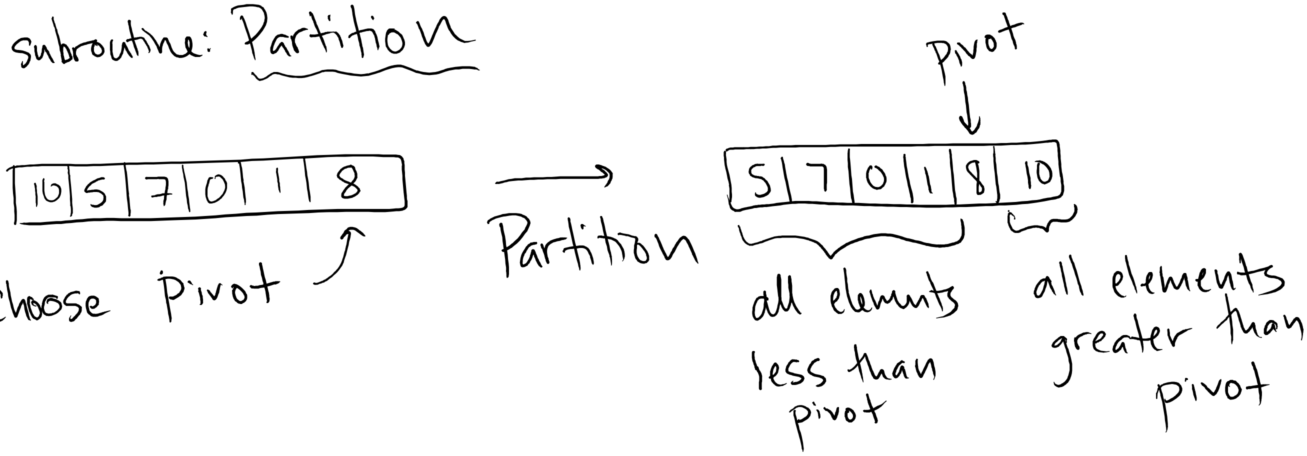
Randomization in Recursive Algorithms

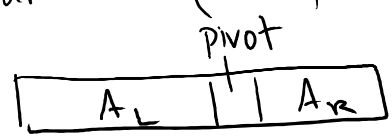
QuickSort Review

Key subroutine: Partition



Input: Array A of length n, no repeated elements
Output: Array with sorted elements

QuickSort(array A)

1. If $|A| = 1$: return A
2. pivot = ChoosePivot(A)
3. Partition(A, pivot)
4. 

5. QuickSort(A_L)
6. QuickSort(A_R)

Conquer

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Partition(A, p)

A:

10	5	12	0	1	8

p = value of pivot

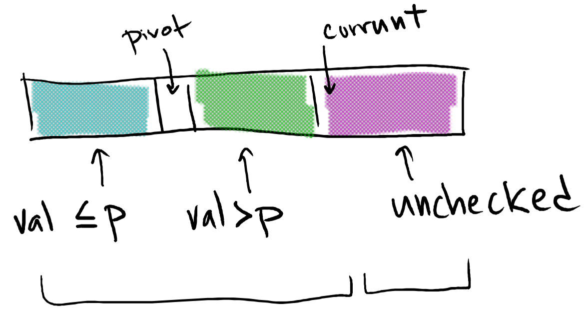
(0) If size = 1, return

(1) Move pivot to start

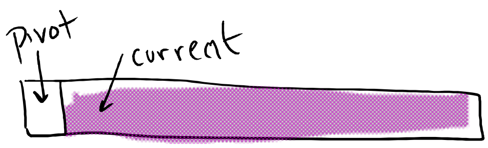
8	5	12	0	1	10

(2) Loop invariant:

Array looks like



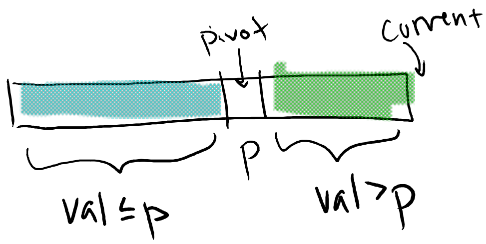
Base Case:



Each step:

- compares current to pivot
- does swaps to maintain invariant ✓
- increases current

Termination:



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Runtime of Quicksort is $O(\#\text{ of comparisons})$

Pf: Partition does most of the work, and runtime of Partition is $O(\#\text{ of comparisons.})$

Q: How many comparisons are done by Partition on input array of size n?

A: $O(\sqrt{n})$ B: $O(n)$ C: $O(n\log n)$ D: $O(n^2)$

Q: What is the runtime of QuickSort when the pivot is always chosen to be $\left(\frac{n}{2}\right)^{th}$ largest element of array?

A: $O(\sqrt{n})$ B: $O(n)$ C: $O(n\log n)$ D: $O(n^2)$