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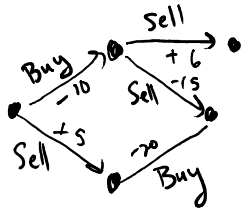
Dijkstra

Good: Fast! $O(m \log n)$ run time

Bad:
- Need to maintain global heap (impossible for internet)
- Fails with negative weights (see HW)

Financial Transactions:



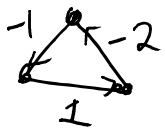
Bellman-Ford

(actually used for internet routing!)

- Dynamic Programming
- Slower but no global heap, negative weights ok.

Bellman Ford:

Input: directed graph $G=(V,E)$, edge weights ℓ_e, vertex $s \in V$, assume no negative cycles



← Neg. cycle

Output: Shortest paths from s to all other $v \in V$

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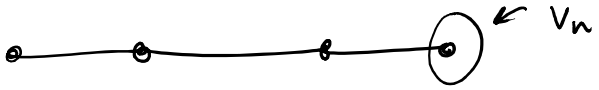
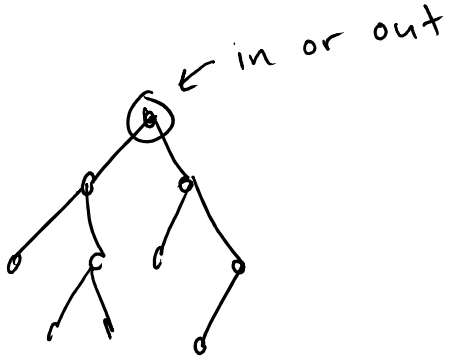
To Create D.P. (dynamic programming) algorithm:

1. Think of options for one (final) piece of optimal solution
2. Write optimal solution in terms of optimal solution to subproblem:

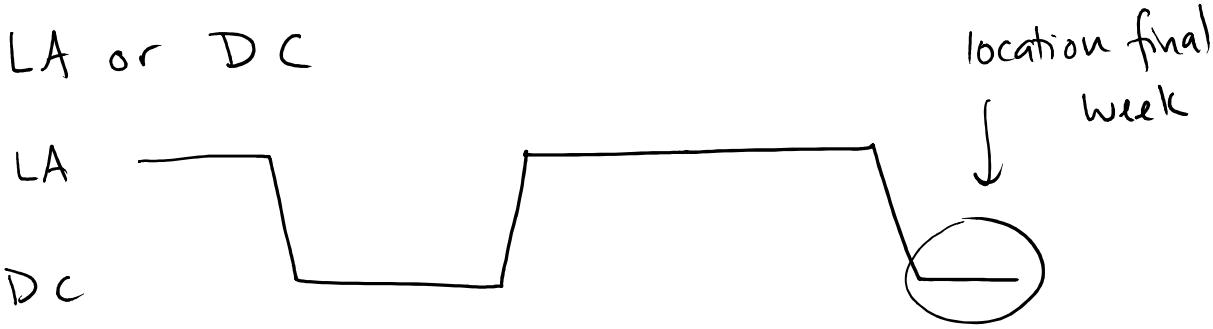
- WMIS on line: $v_n \in S$ (ii) or $v_n \notin S_n$ (i)



- WMIS on tree



- LA or DC



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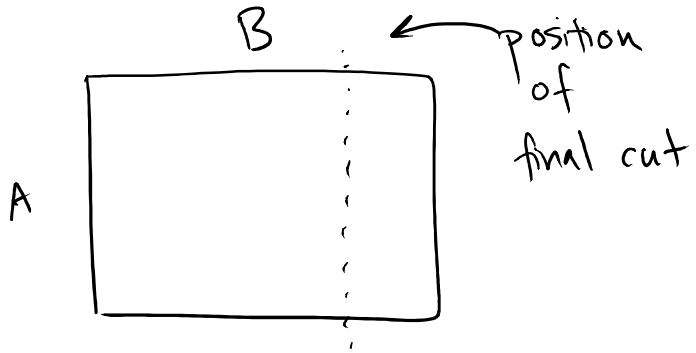

3. Create recurrence relation for objective function
4. Fill in array. START from base case! Go in opposite direction of recurrence
5. Work backwards to find optimal solution if desired.

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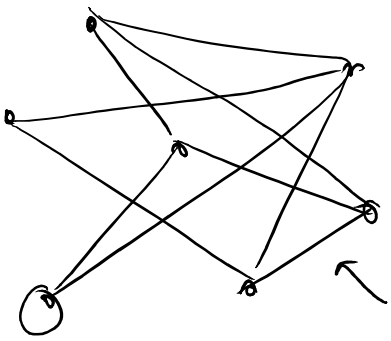
Problem: on general graphs, hard to define subproblems

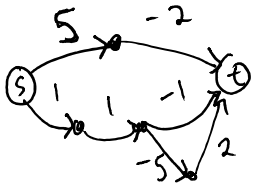


Idea: Edge present or not?

- Why this edge?
- Solution could change wildly if edge present
- Not simple recursive relationship

Same for removing vertex

Q:



What is shortest path from s to t with at most 2 edges? at most 3 edges?

A) 3, 1

B) 2, 0

C) 3, -1

D) 2, 1

We'll use max # of edges in path to order our subproblems

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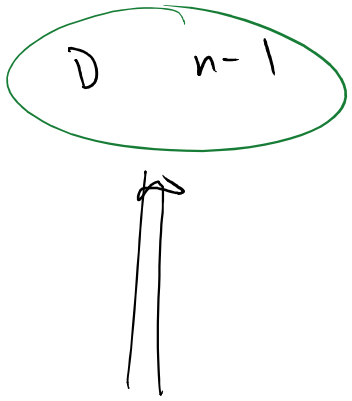
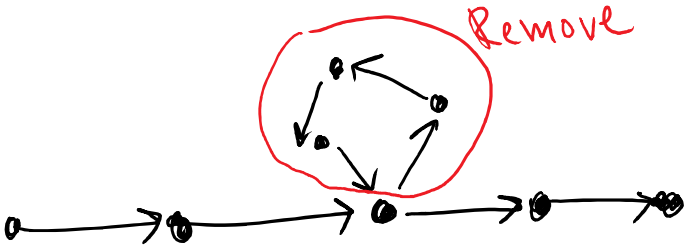
Q: If a graph G has no negative weight cycles, how many edges are in the shortest path. (Let $|V|=n$, $|E|=m$

A) no bound B) m C) n D) n-1



Proof:
- For contradiction, suppose the shortest path has more than n-1 edges.
- Then path must visit same vertex twice, so it contains a cycle
- All cycles have non-negative weight, so if remove cycle from path, the result is a shorter path, a contradiction.



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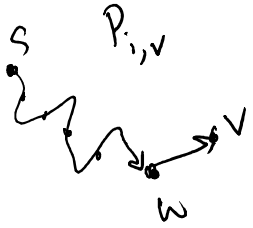
Let $P_{i,v}$ = shortest $s-v$ path with at most i edges
(∞ if no $s-v$ path) (assume unique $\forall\ v, i$)

Q: Fill in blue

Case 1: If $P_{i,v}$ has $\leq (i-1)$ edges,
$P_{i,v} = P_{i-1,v}$

Case 2: If $P_{i,v}$ has i edges, then
$P_{i,v} = P_{i-1,w} + (w,v)$
for some $w: (w,v) \in E$



To prove, use proof by contradiction:

- Case 1: Suppose for contradiction $P_{i,v}$ has less than i edges, but $P_{i,v} = P^* \neq P_{i-1,v}$.
Then P^* is a path with at most $i-1$ edges from s to v. But $P_{i-1,v}$ is the shortest such path, so $\ell(P_{i-1,v}) \leq \ell(P^*)$, so you can always choose $P_{i,v} = P_{i-1,v}$ to get shortest possible path

Case 2: Assume for contradiction $P_{i,v}$ has i edges, but is $P^*_w + (w,v)$ for $P^*_w \neq P_{i-1,w}$.

...

(rest is similar to Case 1)

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Q: How many subproblems must be evaluated to calculate $P_{i,v}$?

A) $n+1$ B) n C) $1 + |\{u : (u,v) \in E\}|$ D) $|\{u : (u,v) \in E\}|$

Case 1: $P_{i-1,v}$ (1 subproblem)

Case 2: $P_{i-1,w}$, for each $w \in \{u : (u,v) \in E\}$ ($|\{u : (u,v) \in E\}|$ subproblems)

* Cycles are allowed – limit on i keeps from infinite cycles

3 (Dynamic Programming) Create recurrence relation

Let $L_{i,v}$ be length of path $P_{i,v}$ (∞ if no path)

Q. Base Case: $L_{0,s} = 0$ $L_{0,v} = \infty \ \forall \ v \in V - s$

Recurrence: $L_{i,v} = \min \begin{cases} L_{i-1,v} \\ \min_{(w,v) \in E} L_{i-1,w} + \ell_{wv} \end{cases}$

Correctness: Using proof on previous page, $P_{i,v}$ must be related to one of $1 + |\{w : (w,v) \in E\}|$ subproblems. We look at all (exhaustive search)

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Q:

Pseudo Code :

Note: Start at Base case!

$L[i, s] = 0 \quad \forall \; i$

$L[0, v] = \infty \quad \forall \; v \in V - s$

for $(i = 1$ to $n-1)\{$ ← max # edges required is n-1 if no neg. cycles (see HW)

for $(v \in V - s)$

$$L[i, v] = \min \{ L[i-1, v], \min_{(w,v) \in E} L[i-1, w] + l(w, v) \}$$

- Assume have inverse adjacency list $A'_G[v] = \{ u : (u, v) \in E \}$
- Do for loop over $A'_G[w]$

$\}$

Q: What is runtime of Bellman Ford? (Pick strongest bound)

A) $O(n^2)$ B) $O(mn)$ C) $O(n^3)$ D) $O(m^2)$

$$\sum_{i=1}^{n-1} \sum_{v} \left(1 + \sum_{(w,v) \in E} 1 \right)$$

$$n(n + m) = O(nm)$$

($n \leq m$ if G is connected)