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Michael
Cze.
Jackson
Joonwoo

S. KIMMEL

Shortest Paths

E = set of edges
V = set of vertices

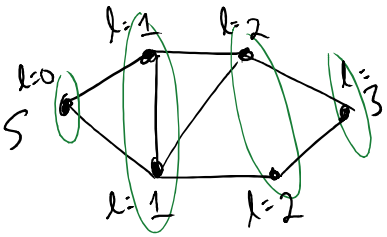
Input: Unweighted, undirected Graph $G=(V,E)$, starting node $s \in V$

Output: Array l, $l[v]$ = length of shortest path from s to v.

($l[v] = \infty$ if no path from s to v)

Applications? Maps, social connections, finacial transactions

Idea: explore each distance layer in turn:



Similar? Breadth-First Search!

```
l[v] = ∞  ∀ v ∈ V
vis[v] = false  ∀ v ∈ V      //make true after visiting
A = {}                       // create queue
A.add(s)                     // initialize with vertex s
l[s] = 0
vis[s] = true
while (A is not empty)
    - v = A.pop
    - for w: {v,w} ∈ E :
        if (vis[w] = false):
            • A.add(w)
            • Vis[w] = true
            • l[w] = l[v] + 1.
```

If remove lines involving l, have BFS

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Q: Will the algorithm be correct under following conditions?
(If not always, provide counter example.)

- directed
- positive weighted edges
- negative weights (uniform)

- What is time complexity of graph search if you store G in adjacency list data structure?

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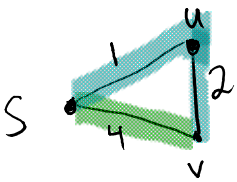
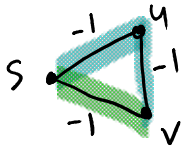
Q: Will the algorithm be correct under following conditions?
(If not always, provide counter example.)

- directed — OK!
- positive weighted edges — No



BFS Solution: $\ell[v]=4$
Correct: $\ell[v]=3$

- negative weights (uniform) — No



BFS Solution $\ell[v]=-1$
Correct: $\ell[v]=-2$

- What is time complexity of graph search if you store G in adjacency list data structure?

$O(n+m)$ $\quad n=|V| \quad m=|E|$

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Explain: Why is runtime $O(n+m)$

- Looks at edges (in for loop)
- Each edge can only be examined when its adjoining vertex is popped. Each vertex can only show up in QUEUE one time. $\Rightarrow$ Each edge is examined twice

- Looking at each edge takes constant time b/c use adjacency list.

$O(m)$ to do while loop

+

Initialization: $O(n)$

$O(n+m)$ $= O(n+m)$

Very fast!

(Optimal! Why?)

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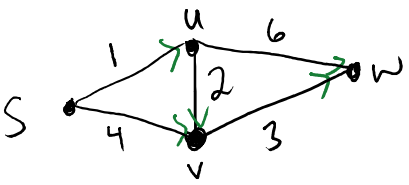
Shortest Paths



Input: • Directed graph $G = (E, V)$ with positive weights l_{uv} $\forall$ $(u,v) \in E$

• vertex $s \in V$

Output: $\forall$ $v \in V$, find

$L(v)$ = length of shortest path from s to v

- path following direction of edges
- add weights of edges on path

Q: What is $L(s), L(u), L(v), L(w)$ for graph above?

A) $0, 1, 4, 6$ B) $0, 1, 3, 7$ C) $0, 1, 1, 2$

D) $0, 1, 3, 6$

Applications: navigation
financial transactions

Assumptions:
- s is connected to all other vertices (not important, just makes life easier)
- No negative edge weights (Important!)

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Dijkstra's Algorithm : Intuition, BFS

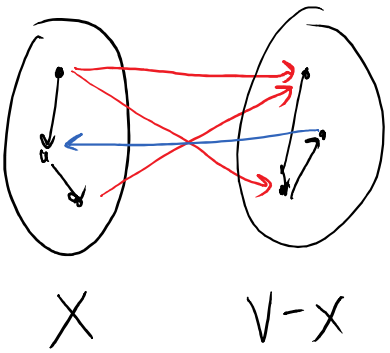
Initialization:

$X = \{s\}$ (vertices processed)

$A[s] = 0$ (array storing shortest path distance to v from s)

$B[s] = \phi$ (array storing shortest path to v from s)

B not necessary for implementation, just helpful for understanding



We care about edges from X to V-X

While $X \neq V$:

- among edges $(v,w) \in E$ with $v \in X$, $w \in V-X$, pick edge that minimizes

$$A[v] + \ell_{vw}$$

(ℓ_{vw}: weight of edge (v,w))

Dijkstra's greedy criterion

Let (v^*, w^*) be minimizing edge

- $X = X + w^*$
- $A[w^*] = A[v^*] + \ell_{v^*w^*}$ ($A[v^*]$: already computed, since $v^* \in X$)
- $B[w^*] = B[v^*] + (v^*, w^*)$

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Q: What kind of algorithm is Dijkstra's Algorithm?

A) Divide & Conquer

B) Dynamic Programming

No recurrence relation

C) Greedy

- No smaller subsystems
- Choose best thing right now
- Easy to describe
- Hard to prove

D) Other

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Runtime with Heap + Adjacency List

Run Time

$X = \{s\}$
$A[s] = 0$
$B[s] = \phi$

→ $O(n)$ to initialize length n arrays

Each $v \in V - s$ calculate $k(v) = \ell_{sv}$ store in heap

Initialize heap

→ $O(n \log n)$

While $X \neq V$

- among edges $(v, w) \in E$ with $v \in X$, $w \in V - X$, pick edge that minimizes

  $A[v] + \ell_{vw}$

→ $O(1)$ ⇒ total $O(n)$

Let (v^*, w^*) be minimizing edge

- $X = X + w^*$
- $A[w^*] = A[v^*] + \ell_{v^* w^*}$
- $B[w^*] = B[v^*] + (v^*, w^*)$
- Update heap
  - Remove w^* → $O(\log n)$ ⇒ Over algorithm: $O(n \log n)$
  - Update keys → Tricky - see next page