CS 260 - Equivalence relations

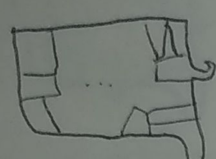
learning nuggets:

1. say (in words) what is a relation
2. describe three properties of relations and how this gives us equivalence relations
3. describe how equivalence classes partition a set.
4. prove whether (or not) a relation is an equivalence relation.

what we want: divide a set into subsets

why: might be easier to work with (write algorithm, prove) these subsets instead of full set.

ex1: cities and states



A = all cities

B = states

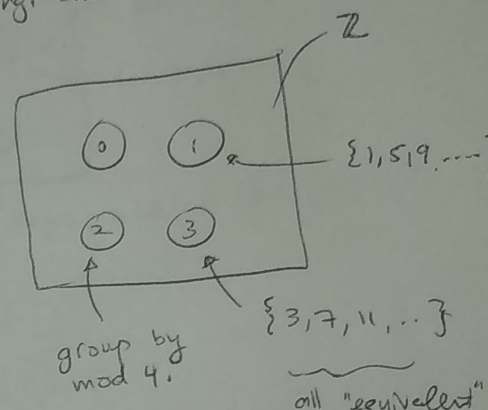
$A \times B = \{(a_1, b_1), (a_2, b_1) \dots\}$

↑ cartesian product   ↑ order matters!

say, related if city is in state so (Boston, MA) related but (Boston, VT) not related.

→ related city-states subset of $A \times B$

ex2: congruence modulo 4



how do we define these groups?

↓

equivalence relations.

def: let A be a set. A relation R on A is a subset of $A \times A$.

$$\boxed{R \subseteq A \times A}$$

if $x, y \in R$ we say xRy "x is related to y".

ex1: $R = \{(x,y) : x < y\}$

a) $(1,2) \in R$ ← order matters!

b) $(2,1) \notin R$

c) $(1,1) \notin R$

ex2: $R = \{(x,y) : x \leq y\}$

now $(1,1) \in R$ → special property!

what are other properties?

properties of relations: let $R \subseteq A \times A$. (A is a set).

① **reflexive:** $(a,a) \in R$ or aRa $\forall a \in A$ — **example:** $\leq$

② **symmetric:** if $(a,b) \in R$ then $(b,a) \in R$ $\forall a,b \in A$ — **example** $ab \geq 1$, $ba \geq 1$

③ **transitive:** if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$. $\forall a,b,c \in A$ — **example:** $<$ — $a < b$, $b < c$, $\therefore a < c$

now we can define
equivalence relations: relations which are **reflexive**, **symmetric** and **transitive**

① how to prove if it is? check three properties

② how to prove if it is not? counterexample.

example: congruence modulo m

$$R = \{ (a,b) \mid a \equiv b \pmod m \}$$

"m divides a-b" ← say in words!

reflexive? $a - a = 0 = (0)m$, $0 \in \mathbb{Z}$ ✓

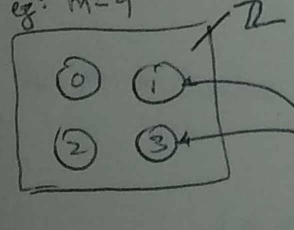
symmetric? $a - b = km$ what about $b-a$? $b-a = -(a-b) = lm = (-k)m$, $-k \in \mathbb{Z}$ ✓

transitive? if $(a,b) \in R \wedge (b,c) \in R$, $(a,c) \in R$?

$$\left.\begin{aligned} a-b &= km \\ b-c &= lm \end{aligned}\right\} \text{add}$$

$$a - c = \underbrace{(k+l)}_{\in \mathbb{Z}} m \quad ✓$$

eg: $m=4$

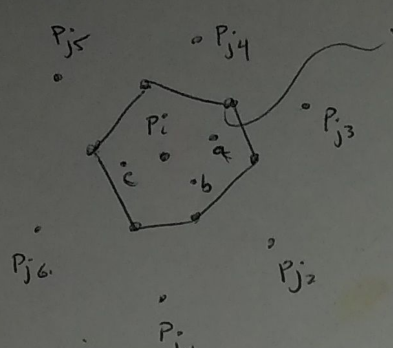


equivalence classes if a pair $(a,b) \in R$, $a \equiv b$ means a,b are in same equivalence class.

* given an equivalence relation R on set A. The equivalence classes of R form a **partition** of A. $\{A_i \mid i \in I\}$

we can go the other way given a partition of A, $\exists$ equivalence relation R that has sets $A_i, i \in I$ as equivalence classes.

Voronoi example: does this define an equivalence relation?



V_i → we partioned B (blackbord points) into partitions $\{V_i \mid i \in [1,\dots,m]\}$

reflexive? $(a,a) \in R$? a is in the same subset as itself!

symmetric? $\underbrace{(a,b)}_{\text{in the region } V_i} \in R \rightarrow \underbrace{(b,a)}_{\text{also in the region}} \in R$ ✓

transitive? $\underbrace{(a,b) \in R}_{\text{partition } V_i} \wedge \underbrace{(b,c)}_{\text{partition } V_j} \in R$ then is $(a,c) \in R$,

so be belongs to V_i and V_j
then $V_i = V_j$ and $(a,c) \in R$

how to show if something is not an equivalence relation?

eg. $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \quad a|b\}$ reflexive, symmetric, transitive?

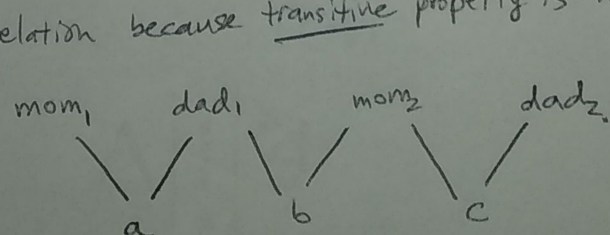
$2|4$ but $4 \nmid 2$
prove by counterexample.

more examples: • decide if equivalence relation (or not) and why
• if equivalence relation, what are equivalence classes?
S = set of all people who ever lived.

1.) $R \subseteq S \times S$, $R = \{(a,b)$: a,b have same parents$\}$

2) $R \subseteq S \times S$, $R = \{(a,b)$: a,b share a parent$\}$

solution:
1. equivalence relation → equivalence classes are groups of full siblings.
2. not equivalence relation because transitive property is not true.



we'll see relations again when we talk about functions
(f from A to B is a subset of $A \times B$.)