S.KIMMEL

Goals

- Notice style of proofs
- Prove using direct + contrapositive approaches

Q: Which of the following is logically equivalent to

$$\neg(\forall x \in S, P(x))$$

A) $\forall x \in S, \neg P(x)$

B) $\exists x \in S : P(x)$

C) $\neg(\exists x \in S : \neg P(x))$

D) $\exists x \in S : \neg P(x)$

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C) $\neg(\exists x \in S : P(x))$

D) $\exists x \in S : \neg P(x)$

ex: $S = \mathbb{Z}$, $P(x) \equiv x^2 = x$

$\neg(\forall x \in S, P(x)) \equiv$

Not every integer is its own square

$\exists x \in S, \neg P(x)$

There is an integer that is not its own square.

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Q: Which of the following is logically equivalent to

$$\neg\left(\exists x \in S,\ P(x)\right)$$

A) $\forall x \in S, \neg P(x)$

B) $\exists x \in S: P(x)$

C) $\neg\left(\forall x \in S, \neg P(x)\right)$

D) $\exists x \in S: \neg P(x)$

Q: Which of the following is logically equivalent to

$$\neg(\exists x \in S, P(x))$$

A) $\forall x \in S, \neg P(x)$ $\leftarrow$

B) $\exists x \in S, P(x)$

C) $\neg(\forall x \in S, \neg P(x))$

D) $\exists x \in S, \neg P(x)$

ex: $S = \mathbb{N}$, $P(x) \equiv x = -x$

$\neg(\exists x \in S, P(x)) \equiv$

No natural number is equal to its negation

$\forall x \in S, \neg P(x)$

Every natural numbe is not equal to its negation

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De Morgans Rules

Logically equivalent to

$\forall \longleftrightarrow \exists$ & bring negation inside quantifier

$$\neg\forall, P(x) \equiv \exists, \neg P(x)$$

$$\neg\exists, P(x) \equiv \forall, \neg P(x)$$

For example:

double negative doesn't change

de Morgan

$$\forall x \in S, P(x) \equiv \neg\neg \forall x \in S, P(x) \equiv \neg \exists x \in S : \neg P(x)$$

CS200 - Worksheet 2

(Taken from *Discrete Mathematics, an Open Introduction* by Levin). For each of the following proofs, translate the statement that is proved into math. Use $m|n$ to be the predicate m divides n. Next discuss the language. What words are used repeatedly, and what do those words signal to the reader? What do you notice about the style? Anything else you notice?

1. Suppose a and b are odd. That is, $a = 2k + 1$ and $b = 2m + 1$ for some integers k and m. Then

$$\begin{aligned} ab &= (2k + 1)(2m + 1) \\ &= 4km + 2k + 2m + 1 \\ &= 2(2km + k + m) + 1. \end{aligned} \tag{1}$$

   Therefore, ab is odd.

2. Assume that a or b is even. Suppose it is a, since the case where b is even will be identical. That is, $a = 2k$ for some integer k. Then

$$ab = (2k)b = 2(kb). \tag{2}$$

   Therefore ab is even.

3. Suppose that ab is even but a and b are both odd. Namely, $a = 2k + 1$ and $b = 2j + 1$ for some integers k, and j. Then

$$\begin{aligned} ab &= (2k + 1)(2j + 1) \\ &= 4kj + 2k + 2j + 1 \\ &= 2(2kj + k + j) + 1. \end{aligned} \tag{3}$$

   But this means that ab is odd, which contradicts our premise. Thus a and b can not both be odd.

4. Assume ab is even. Namely, $ab = 2n$ for some integer n. Then there are two cases: a must be either even or odd. If it is odd, then $a = 2k + 1$ for some integer k. Then we have

$$\begin{aligned} 2n &= (2k + 1)b \\ &= 2kb + b. \end{aligned} \tag{4}$$

   Subtracting $2kb$ from both sides, we get

$$2(n - kb) = b. \tag{5}$$

   Therefore, b must be even. The other case is that a is even, so we find that either a or b is even.

Solution

1. $\forall a, b \in \mathbb{Z}, (\neg 2|a) \wedge (\neg 2|b) \rightarrow (\neg 2|ab)$. This is the same as statement 4, but using proof by contrapositive.

2. $\forall a, b \in \mathbb{Z}, 2|a \vee 2|b \rightarrow 2|ab$. This proof is doing something different from the others. It proves the converse of 4.

3. $\forall a, b \in \mathbb{Z}, 2|ab \wedge \neg(\neg 2|a) \wedge (\neg 2|b)) \rightarrow ((\neg 2|a) \wedge (\neg 2|b))$. This is the same as statement 4, but using proof by contradiction.

4. $\forall a, b \in \mathbb{Z}, 2|ab \rightarrow 2|a \vee 2|b$

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Discussion of Proof Language

- Many ways to prove the same statement is true. Proof writing is art & science.
- "Suppose" "Assume" signals premise.
- Periods at end of equations
- Each new symbol is explained "for some integer"
  - ⇒ Just like any new symbol should be quantified
- "Then", "Therefore," "Thus" signals deduction
- "That is," "Namely", signals a definition or explanation
-

☆ Look for these words in other proofs

☆ Use in your own proofs.