CS200 - Problem Set 2

Please read the sections of the syllabus on problem sets and honor code before starting this homework.

1. *Inductive Proofs*

   (a) **[11 points]** Prove using induction that for $n \geq 0$, $7^n - 2^n$ is divisible by 5. (An integer m is divisible by an integer r if $m = r \cdot g$, where g is some other integer.)

   (b) **[11 points]** Prove using induction that $2^n > n^2$ whenever n is an integer, and $n \geq 5$.

   (c) **[11 points]** Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for any integer n such that $n \geq 1$.

   (d) **[11 points]** Finish the following proof that Algorithm 1 correctly multiplies an integer $n \geq 0$ and an integer b.

   Algorithm 1: `Mult`(n, b)

   ```
   Input  : Non-negative integer n, and integer b
   Output: n × b
   /* Base Case                                   */
   1 if n == 0 then
   2 |   return 0;
   3 else
     |   // Recursive step
   4 |   return b + Mult(n − 1, b);
   5 end
   ```

   Proof: Let $P(n)$ be the predicate: `Mult`(n,b) correctly outputs the product of n and b. We will prove using induction that $P(n)$ is true for all $n \geq 0$.

   For the base case, let $n = 0$. In this case, we see the `If` statement is true at line 1, and so the algorithm returns 0. This is precisely what we want, since $0 \times b = 0$ for any integer b, so the algorithm is correct and $P(0)$ is true.

   For the inductive step...

2. **[3 points each]** For each of the following sentences, decide whether it is a statement, predicate, or neither, and explain why

   (a) Call me Ishmael.

   (b) The world is supported on the back of a giant tortoise.

   (c) x is a multiple of 7.

(d) The next sentence is true.

(e) The preceding sentence is false.

(f) The set $\mathbb{Z}$ contains an infinite number of elements.

3. [**3 points each**]
There are many ways to represent the logical implication $(P \to Q)$ in English. To make proofs more interesting to read, we often take advantage of these different ways of phrasing the same underlying mathematical statement. In the following, I will ask you to rewrite sentences in the form $p \to q$. For example, "I get a brain freeze if I eat ice cream" should be rewritten "I eat ice cream $\to$ I get a brain freeze." Normally there are two clear possibilities: $p \to q$ or $q \to p$ and only one of them makes sense. If you are having trouble, check out p. 43 of Book of Proof, or problem 5 in Chapter 0 of DMOI (which has solutions).

   (a) I open my umbrella whenever it rains.

   (b) I miss class only if I am unwell.

   (c) You can't invent unless you are curious and knowledgeable.

4. [**6 points**] Prove using a truth table that $((A \vee B) \wedge (A \to C) \wedge (B \to C)) \to C$ is true. (Note $P \wedge Q \wedge R$ is only true when all of the predicates are true, and is false otherwise.) This statement is also known as "proof by cases." Please explain why. (Hint: the two cases are related A and B, and we want to say something about C.)

5. How long did you spend on this homework?