# Announcements

- Please bring laptops on Wednesday and Friday. (Can check out from Davis.)
- Programming Assignment due Tuesday
- Overall reflection due Thursday (see instructions)
- Exam Content: Fully cumulative. Emphasis on most recent topics.
- 2-sided cheat sheet – no other resources

# Review Topics

- Recurrence Relations (strings)
- Recurrence relations (pseudocode)
- Indicator Random Variables
- Graph Pseudocode
- Equivalence Relations

# Recurrence Relations

Let $T(n)$ be the number of strings in $\{0,1,2\}^n$ that do NOT contain two consecutive zeros. Write a recurrence relation for $T(n)$.

**Algorithm 1:** `MergeSort`$(C, n)$

**Input** : Array of $C$ of length $n$ (where $n$ is a power of 2)
**Output:** Sorted array containing all elements of $C$

```
if n==1 then
    return C;
end
A=MergeSort(C[1 : n/2], n/2);
B=MergeSort(C[n/2 + 1 : n/2], n/2);
p_A = 1;
p_B = 1;
Increase length of A and B by 1 each, and set final element of each array to ∞;
for k = 1 to n do
    if A[p_A] < B[p_B] then
        C[k] = A[p_A];
        p_A+ = 1;
    else
        C[k] = B[p_B];
        p_B+ = 1;
    end
end
```

- Create a recurrence relation for the runtime of MergeSort, and evaluate using the iterative method and the tree method (use formula).

# Indicator Random Variables

Consider an ordered list containing the elements $\{1,2,3,\ldots,n\}$ with no repeats. An inversion is a pair $(i,j)$ where $i < j$ but $j$ precedes $i$ in the list. For example if we consider the ordered list $(3,1,4,2)$ of the elements there are 3 inversions: $\{1,3\}, \{2,3\}, \{2,4\}$.

If an ordering is chosen with equal probability from among all possible orderings, what is the average number of inversions?

1. What is the sample space and random variable of interest?
2. Write rand. variable as weighted sum of indicator rand. variables
3. Use linearity of expectation and property of indicator rand. variables to calculate the expectation.

# Graph Pseudocode

Given a graph $G = (V, E)$ and a vertex $v \in V$, write pseudocode to determine if $G$ is a star centered at $v$. A star is a graph with one central vertex that is connected to every other vertex, but aside from the central vertex, no other two vertices are connected. You can choose either Adjacency Matrix or Adjacency List…but think about which is easier before you start! What is big-O runtime of your algorithm? (For extra practice, write code using the other graph data structure.)

# Equivalence Relation(?)

Either prove it is an equivalence relation, or prove it is not an equivalence relation. If it is, describe the equivalence classes

Let $R \subseteq \mathbb{Z} \times \mathbb{Z}$, where $R = \{(a, b): 2 | (a - b)\}$

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# Recurrence Relation Solution

Let $T(n)$ be the number of strings in $\{0,1,2\}^n$ that do NOT contain two consecutive zeros. Write a recurrence relation for $T(n)$.

3 options: _ _ _ _ _ 1 or _ _ _ _ _ 2 or _ _ _ _ _ 0

- If end in 1 or 2, need there to not be consecutive zeros in the first n-1 positions. There are $T(n-1)$ ways of doing this for each.
- If end in 0, we need the second to last digit to be a 1 or 2. Otherwise we have two consecutive 0's. Then for the remaining n-2 digits, we need there to be no two consecutive 0's. There are $T(n-2)$ ways of doing this. So using the product rule, there are $2T(n-2)$ ways.
- Using the sum rule: $T(n) = 2\left(\mathrm{T}(\mathrm{n}-1) + \mathrm{T}(\mathrm{n}-2)\right)$

Need two base cases to avoid falling off the ladder: $T(1) = 3, T(2) = 8$.

# Recurrence with an Algorithm Solution

**Algorithm 1:** MergeSort$(C, n)$

**Input :** Array of $C$ of length $n$ (where $n$ is a power of 2)

**Output:** Sorted array containing all elements of $C$

```
if n==1 then
    return C;
end
A=MergeSort(C[1 : n/2], n/2);
B=MergeSort(C[n/2 + 1 : n/2], n/2);
p_A = 1;
p_B = 1;
Increase length of A and B by 1 each, and set final element of each array to ∞;
for k = 1 to n do
    if A[p_A] < B[p_B] then
        C[k] = A[p_A];
        p_A+ = 1;
    else
        C[k] = B[p_B];
        p_B+ = 1;
    end
end
```

- Create a recurrence relation for the runtime of MergeSort, and evaluate using the iterative method.

- $T(n) = O(n) + 2T\left(\frac{n}{2}\right), T(1) = O(1)$
- Using iterative method, pattern is $T(n) = 2^k T\left(\frac{n}{2^k}\right) + O(nk)$. Plug in $k = \log_2 n$ to get $T(n) = n + O(n \log_2 n) = O(n \log_2 n)$

# Indicator Random Variables Solution

1. What is the sample space and key random variable?
   - Sample space is set of $n!$ possible permutations of $n$ elements. Random variable $X$ is the number of inversions in a single permutation
2. Break key random variable into sum of indicator random variables
   - Let $X_{ij}$ take value $1$ if there is an inversion between $i, j$, and $0$ else. Then $X = \sum_{ij} X_{ij}$, where the sum is over all unordered pairs of vertices where $i \neq j$.
3. $E[X] = \sum_{ij} E[X_{ij}]$ (using linearity of expectation)

$E[X] = \sum_{ij} \Pr[\text{there is an inversion of } i, j]$. (using property of ind. rand. vars.)

Any two elements are equally likely to be inverted or not! So

$\Pr[\text{there is an inversion of } i, j] = \frac{1}{2}$. And hence

$$E[X] = \sum_{ij} 1/2 = \frac{C(n,2)}{2} = n(n-1)/4$$

# Graph Pseudocode Solution

Input: Adjacency List A, vertex v:

- If (length of $A[v]$) $\neq |V| - 1$: return False.
- For $u \in (V - \{v\})$:
  - If (length of $A[u]$) $\neq 1) \vee (A[u, 1] \neq v)$: return False
- Return True

Runtime: O(|V|)

# Equivalence Relation(?)

- Let $R \subseteq \mathbb{Z}\times\mathbb{Z}$, where $R = \{(a, b): 2|(a - b)\}$
  - Reflexive: Let $a \in \mathbb{Z}$. Then $(a - a) = 0$, and $2|0$, since $2\times0 = 0$.
  - Symmetric: Let $a, b \in \mathbb{Z}$. Assume $(a, b) \in R$. Then $\exists c \in \mathbb{Z}: 2c = a - b$. But then $2(-c) = b - a$, so $2|(b - a)$, and so $(b - a) \in R$.
  - Transitive. Let $a, b, c \in \mathbb{Z}$. Assume $(a, b) \in R$ and $(b, c) \in R$. Then $\exists f, d \in \mathbb{Z}: (2f = a - b) \wedge (2d = b - c)$. Adding these two equations, we get $2(f + d) = a - c$, so $(a, c) \in R$.
  - Thus it is an equivalence relation. An equivalence classes is the set of even numbers.
- Let $R \subseteq \mathbb{Z}\times\mathbb{Z}$, where $R = \{(a, b): (a - b)|2\}$. Not transitive: $(3,1) \in R$ and $(5,3) \in R$,but $(5,1) \notin R$.